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Basics of Integration

① $\int x^n dx = \frac{x^{n+1}}{n+1}$

② $\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2}$

③ $\int \frac{1}{x} dx = \log x$

④ $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$

⑤ $\int \frac{1}{x^n} = \frac{-1}{(n-1)x^{n-1}}$

⑥ $\int e^{ax} dx = \frac{e^{ax}}{a}$

⑦ $\int a^x dx = \frac{a^x}{\log a}$

⑧ $\int \frac{1}{(ax+b)} dx = \frac{\log(ax+b)}{a}$

* ⑨
$$\int \begin{cases} \sin x = -\cos x \\ \cos x = \sin x \\ \sec^2 x = \tan x \\ \sec x \tan x = \cancel{\cot}\ \sec x \\ \operatorname{cosec} x \cot x = -\operatorname{cosec} x \\ \operatorname{cosec}^2 x = -\cot x \end{cases}$$

⑩ $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

(11) $\int \frac{1}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$

(12) $\int \frac{1}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

(13) $\int \frac{1}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)$

(14) $\int \frac{1}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right)$

* (15) $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

(16) # $\int \sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

(17) $\int \sqrt{x^2-a^2} = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

(18) $\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

Nr is derivative of Dr

(19) ## $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$

(20) # $\int \tan x\,dx = \log(\sec x)$

(21) # $\int \cot x\,dx = \log(\sin x)$

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(22) * $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$

(23) * $\int [f(x)]^n f'(x) dx = \frac{f(x)^{n+1}}{n+1}$

Ex: $\int \sqrt{x^2+a^2} \cdot x\, dx = \frac{1}{2}\int \sqrt{x^2+a^2} \cdot 2x\, dx$

$= \frac{1}{2} \frac{(x^2+a^2)^{3/2}}{3/2}$

$= \frac{(x^2+a^2)^{3/2}}{3}$

Integrals of the form:- $\int e^{f(x)} dx$

If $\int e^{ax+b} dx = \frac{e^{ax+b}}{a}$

In general $\int e^{f(x)} dx$,

Put $f(x) = t$

(24) $\int e^{f(x)} f'(x)\, dx = e^{f(x)}$

(25) $\int \tanh x\, dx = \log(\cosh x)$

(26) $\int \coth x\, dx = \log(\sinh x)$

Integration by u.v Rule:

$\left\{ \int uv\, dx = u\int v\, dx - \int\left[\int v \cdot \frac{du}{dx}\right] dx \right\}$

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Shortcut to UV Rule:

$$\int uv\,dx = uV_1 - u'V_2 + u''V_3 - u'''V_4 + \cdots$$

$$\left[\begin{array}{l} u', u'' = \text{derivative} \\ V_1, V_2 = \text{Integration} \\ u = \text{only algebraic} \end{array}\right]$$

Integration of the type $\sin^n x, \cos^n x$.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$$

$$\int \sin^3 x\,dx = \int \frac{3\sin x - \sin 3x}{4}$$

$$\int \cos^3 x\,dx = \int \frac{\cancel{4\sin}\ 3\cos x + \cos 3x}{4}$$

Integrals of the type $\int \sin^{-1} x$, $\int \cos^{-1} x$, $\int \tan^{-1} x$, $\int \log x\ dx$.

Here, we use uv rule by taking $v = 1$.

$\int e^x [f(x) + f'(x)]\,dx = e^x . f(x)$

$\int \frac{1}{1+e^x}\,dx$ {Multiply & divide by e^{-x}}

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx = -\log(e^{-x}+1)$$

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$\int \frac{1}{1+e^x} dx$ [Multiply & divide by e^{-x}]

$$\int \frac{e^x}{e^x+1} dx = \log(e^x+1)$$

$\int \frac{1}{e^x+e^{-x}} dx$ (Multiply & divide by e^x or e^{-x})

$$\int \frac{e^x}{e^{2x}+1} dx \quad (\text{Put } e^x = t) \Rightarrow \int \frac{dt}{1+t^2}$$

Basics of Definite Integration:

① $\int_a^b f(x)dx = -\int_b^a f(x)dx$

② $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

③ $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

④ $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \Rightarrow f(x)$ is even fn

$= 0 \in f(x)$ is odd fn.

⑤ $\int_0^{2a} f(x)dx = \int_0^a [f(x)+f(2a-x)]dx$

Mix. Formulae :

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x \xleftarrow{\text{derivative}} \sinh x$$

$$\frac{1}{x^n} \xrightarrow{\text{deri}} \frac{-n}{x^{n+1}} \quad // \quad \frac{1}{x^3} = \frac{-3}{x^4}$$

$$\left.\begin{aligned} 2\sin A \cdot \cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A \cdot \sin B &= \sin(A+B) - \sin(A-B) \end{aligned}\right\} \text{Khichdi sine}$$

$$\left.\begin{aligned} 2\cos A \cdot \cos B &= \cos(A+B) + \cos(A-B) \\ 2\sin A \cdot \sin B &= \cos(A-B) - \cos(A+B) \end{aligned}\right\} \text{Clean cos}$$

Check List:

① Rule 3 - Homogeneous D.E.
② Rule 4 - (xy)
③ Exact
④ Rule 1 / Rule 2

$D^3+1=0$	$D^3-1=0$
$(D+1)(D^2-D+1)=0$	$(D-1)(D^2+D+1)=0$

Differential Equation of First Order and First Degree:

* Type I:

Exact Differential Equation:

① $Mdx + Ndy = 0$
$M = f(x,y), N = g(x,y)$

② Check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then D.E is Exact.

③ General Solution:
$$\int_{(y=\text{constant})} M\,dx + \int \begin{bmatrix}\text{terms in } N \\ \text{free from } x\end{bmatrix} dy = C$$

* Alitor $\int Mdx$ is very difficult

$$\int \begin{pmatrix}\text{terms in } M \\ \text{free from } y\end{pmatrix} dx + \int N\,dy = C$$

Exam: $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ // $\frac{\partial N}{\partial y} = \cos x + \cos y + 1$

$$\int_{(y=\text{constant})} (y\cos x + \sin y + y)\,dx + \int 0\,dy = C$$

$$\boxed{y\sin x + x\sin y + xy = C}$$

Type-II

D.E reducible to Exact:
In this method the given D.E will not be exact, we can make them exact by using following rules:

Rule 1: * Check if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ alone

* If it is, then we find, Integrating factor $= e^{\int f(x)dx}$

* Multiply the given D.E by I.F., hence it becomes "Exact"
* Find its General Solution.

Rule 2: * Check if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ alone

* If it is, then find $I.F = e^{\int f(y)dy}$

* Multiply the given D.E by I.F., hence it becomes "Exact"
* Find its General Solution.

Bernoulli's Equation: It is of the type

$$\frac{dy}{dx} + xy = x^n y^n$$

Divide throughout by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{x}{y^{n-1}} = x^n$$

$$\text{Put } \frac{1}{y^{n-1}} = V$$

Type-III

Homogeneous Differential Equation:

A D.E is called a Homogeneous D.E if degree of all terms are same.

$$(x^2+y^2)dx - (x^2-y^2)dy = 0$$

$$I.F = \frac{1}{Mx+Ny}$$

Type-IV

Rule:4

The given D.E of the form:-

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

Ex: $(\sin xy + xy + x^2y^2)y\,dx + [1-x^2y^2]x\,dy = 0$

Every term should have xy term as a variable

$$I.F = \frac{1}{Mx-Ny}$$

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Linear Differential Equation

Linear in 'y'

 # Type-I: It is of the form:-

$$\frac{dy}{dx} + Py = Q$$

 # Note: ① Coefficient of $\frac{dy}{dx}$ should be 1

② $\because$ dy is in Nr, there should be only 1 y term with degree 1.

③ P and Q are function of x / constant.

General Solution has 2 steps:

① $I.F. = e^{\int P dx}$

② G.S. is given by
$$y.(I.F) = \int Q(I.F) dx + C$$

 # Type-II: Linear in "x"

It is of the form:

$$\frac{dx}{dy} + Px = Q$$

Note: ① Coefficient of $\frac{dx}{dy}$ should be 1

② $\because$ dx is in Nr, there should be only 1 x term with degree 1.

③ P and Q are function of y / constant.

G.Solution has two steps:

1. $I.F = e^{\int P dy}$

2. G.S is given by: $x(I.F) = \int Q(I.F)\,dy + C$

D.E Reducible to Linear D.E:

 # Type-4x

$f'(y)\frac{dy}{dx} + Pf(y) = Q$

$\frac{dy}{dx}$: $f(y)$ and $f'(y)$
P and Q fn of x

1. Put $f(y) = V$
2. Diff wrt 'x'

$f'(y)\frac{dy}{dx} = \frac{dV}{dx}$

3. D.E: $\frac{dV}{dx} + PV = Q$

Linear in V

The G.S:

$V(I.F) = \int Q(I.F)\,dx + C$

 # Type-4y:

$f'(x)\frac{dx}{dy} + Pf(x) = Q$

$\frac{dx}{dy}$: $f(x)$ and $f'(x)$
P and Q fn of y

1. Put $f(x) = V$
2. Diff wrt 'y'

$f'(x)\frac{dx}{dy} = \frac{dV}{dy}$

3. D.E: $\frac{dV}{dy} + PV = Q$

The G.S:

$V(I.F) = \int Q(I.F)\,dy + C$

Whenever the coefficient of $\frac{dy}{dx}$ is sum or difference of xy terms

i.e. $(xy + x^2y^n)\frac{dy}{dx} = 0$.

Then the sum can't be solve using $\frac{dy}{dx}$.

Hence, use: $\frac{dx}{dy}$.

Exercise 1.1

Exact Differential Equation

Dec-15

1 $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$

Sol. The above equation is of the form

$$Mdx+Ndy=0$$

$$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$$

$$\frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The D.E is Exact.

It's General Solution:

$$\int\underset{(y=\text{const})}{Mdx}+\int\left[\begin{matrix}\text{terms in } N\\ \text{free from } x\end{matrix}\right]dy=C$$

$$\int\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\int 0\,dy=C$$

$$y\left[x+\log x\right]+x\cos y=C$$

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4 $$\frac{dy}{dx} = \frac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$$

Sol. $(x^2 - x\tan^2 y + \sec^2 y)\,dy = (\tan y - 2xy - y)\,dx$

$(\tan y - 2xy - y)\,dx - (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

It is of the form $Mdx + Ndy = 0$

$\frac{\partial M}{\partial y} = \sec^2 y - 2x - 1$ // $\frac{\partial N}{\partial x} = -2x + \tan^2 y - \sec^2 y$

$= \sec^2 y - 2x - 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The D.E is Exact.

It's General Solution:

$$\int_{y=\text{const}} Mdx + \int \begin{bmatrix}\text{terms in } N \\ \text{free from } x\end{bmatrix} dy = C$$

$\int (\tan y - 2xy - y)\,dx + \int -\sec^2 y\,dy = C$

$x(\tan y - \frac{2yx^2}{2} - y) - \tan y = C$

$x\tan y - x^2 y - xy - \tan y = C$

$\tan y\,(x-1) - xy(x+1) = C$

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7 $\left(1+e^{x/y}\right)dx+\left(1-\frac{x}{y}\right)e^{x/y}dy=0$

Sol. It is of the form $Mdx+Ndy=0$

$$\frac{\partial M}{\partial y}=e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial N}{\partial x}=\left[\left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}+e^{x/y}\left(\frac{-1}{y}\right)\right]$$

$$=e^{x/y}\left[\cancel{\frac{1}{y}}-\frac{x}{y^2}-\cancel{\frac{1}{y}}\right]=e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\therefore\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The D.E is Exact

It's General Solution:

$$\int\left(1+e^{x/y}\right)dx+0\,dy=c$$

$$x+\frac{e^{x/y}}{1/y}=c$$

$$\boxed{x+ye^{x/y}=c}$$

Dec-16

12 $\left[\log\left(x^2+y^2\right)+\frac{2x^2}{x^2+y^2}\right]dx+\left[\frac{2xy}{x^2+y^2}\right]dy=0$

Sol. It is of the form $Mdx+Ndy=0$

$$\frac{\partial M}{\partial y}=\frac{1}{x^2+y^2}\times 2y+2x^2\left(\frac{-1}{x^2+y^2}\right)\times 2y$$

$$=\frac{2y}{x^2+y^2}+\left(\frac{2x\times 2y}{x^2+y^2}\right)$$

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$$\frac{\partial M}{\partial y} = \frac{2y}{x^2+y^2}\left(1-\frac{2x^2}{x^2+y^2}\right)$$

$$\frac{\partial N}{\partial x} = \left[\frac{2xy}{x^2+y^2}\right] dy = \frac{(x^2+y^2)2y - 2xy(2x)}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The D.E is Exact.

$\int M dx$ is tough

$$\int\left[\begin{matrix}\text{term in } M \text{ free} \\ \text{from } y\end{matrix}\right]dx + \int N dy = C$$

$$\int 0\, dx + \int \frac{2xy}{x^2+y^2}\, dy = C$$

$$x \int \frac{2y}{x^2+y^2}\, dy = C$$

$$\boxed{x \log(x^2+y^2) = C}$$

(5) $(x\sqrt{x^2+y^2} - y)dx + (y\sqrt{x^2+y^2} - x)\, dy = 0$

Sol.

$$\frac{\partial M}{\partial y} = x \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} - 1$$

$$\frac{\partial N}{\partial x} = y \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} - 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The D.E is Exact.

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General Solution:

$$\int [x\sqrt{x^2+y^2} - y]\,dx + \int 0\,dy = C$$

$$\int \sqrt{x^2+y^2}\cdot x\,dx - \int y\,dx$$

$$\frac{1}{2}\int (x^2+y^2)^{1/2}\,2x\,dx - yx = C$$

$$\frac{1}{2}\frac{(x^2+y^2)^{3/2}}{3/2} - xy = C$$

$$\frac{(x^2+y^2)^{3/2}}{3} - xy = C$$

Imp

Ex: $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Sol: $\frac{\partial M}{\partial y} = -4x - 4y$, $\frac{\partial N}{\partial x} = -4x - 4y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The D.E is Exact

It's General Solution:

$$\int_{y=\text{const}} M\,dx + \int \left[\begin{matrix}\text{terms in } N \\ \text{free from } x\end{matrix}\right] dy = C$$

$$\int x^2 - 4xy - 2y^2 + \int y^2\,dy = C$$

$$\frac{x^3}{3} - \frac{4x^2y}{2} - 2y^2x + \frac{y^3}{3} = C$$

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$$\# \{ e^{\log_e^a} = a \} \#$$

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Exercise 1.2

Ques 1 $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Sol.

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \quad // \quad \frac{\partial N}{\partial x} = y^3 - 4$$

The D.E is not Exact

Consider

Rule 2.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3y^3 - 6}{y^4 + 2y} = \frac{-3(y^3+2)}{y(y^3+2)}$$

$$= \frac{-3}{y}$$

$$I.F = e^{\int f(y)dy} = e^{\int \frac{-3}{y} dy}$$

$$= e^{-3\int \frac{1}{y} dy} = e^{-3\log y}$$

$$= e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiply the given D.E by $\frac{1}{y^3}$

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}, \quad \frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$$

D.E is Exact Now.

It's General Solution

$$\int \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = C$$

$$\left(y + \frac{2}{y^2}\right)x + y^2 = C$$

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Ques (2) $y(x^2y + e^x)\,dx - e^x dy = 0$

$$M = x^2y^2 + e^x y \quad , \quad N = -e^x$$

$$\frac{\partial M}{\partial y} = 2x^2y + e^x \qquad \frac{\partial N}{\partial x} = -e^x$$

The D.E is not Exact.

Consider, Rule 2

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x - 2x^2y - e^x}{x^2y^2 + e^x y}$$

$$= \frac{-2x^2y - 2e^{-x}}{x^2y^2 + e^x y}$$

$$= \frac{-2(x^2y + e^{-x})}{y(x^2y + e^x)} = \frac{-2}{y}$$

$$I.F = e^{\int f(y)dy} = e^{\int \frac{-2}{y} dy}$$

$$= e^{-2\int \frac{1}{y} dy} = e^{-2\log y} = e^{\log y^{-2}}$$

$$I.F = \frac{1}{y^2}$$

Multiplying I.F to the D.E

$$\left(x^2 + \frac{e^x}{y}\right)dx - \frac{e^x}{y^2}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-e^x}{y^2} \;\Big|\Big|\; \frac{\partial N}{\partial x} = \frac{-e^x}{y^2}$$

The D.E is Exact now.

It's General Solution is:

$$\int\left(x^2 + \frac{e^x}{y}\right)dx + \int 0\,dy = C$$

$$\frac{x^3}{3} + \frac{e^x}{y} = C$$

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Ans (3) $\left(\frac{y}{x}\sec y - \tan y\right)dx - (x - \sec y \cdot \log x)dy = 0$

$\frac{1}{\cos y}$ can be taken common from M.

$\frac{\partial M}{\partial y} = \frac{y \tan y \sec y}{x} + \frac{\sec y}{x} - \sec^2 x$

$= \frac{1}{x}\left(y \sec y \tan y + \sec y\right) - \sec^2 x$

$\frac{\partial N}{\partial x} = -1 + \frac{\sec y}{x}$

The D.E is not exact.

Consider Rule 2

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-1 + \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y - \frac{\sec y}{x} + \sec^2 y}{\left(\frac{y \sec y}{x} - \tan y\right)}$$

$$= \frac{-1 - \frac{y \sec y \tan y}{x} + \sec^2 y}{\frac{y \sec y}{x} - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y \sec x}{x} - \tan y}$$

$$= \frac{-\tan y\left(\frac{y}{x}\sec y - \tan y\right)}{\left(\frac{y}{x}\sec y - \tan y\right)}$$

$= -\tan y$

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$$I.F = e^{\int f(y)\,dy} = e^{\int -\tan y\,dy}$$

$$= e^{-\log \sec y} = e^{\log \sec y^{-1}}$$

$$= \frac{1}{\sec y}$$

Multiply $\frac{1}{\sec y}$ to the D.E

$$\left(\frac{y}{x} - \frac{\tan y}{\sec y}\right)dx - \left(\frac{x}{\sec y} - \log x\right)dy$$

$$\left(\frac{y}{x} - \sin y\right)dx - \left(\frac{x}{\sec y} - \log x\right)dy$$

$$\frac{\partial M}{\partial y} = \frac{1}{x} - \cos y \;//\; \frac{\partial N}{\partial x} = \frac{1}{x} - \cos y$$

The D.E is exact now.

Its general solution:

$$\int\left(\frac{y}{x} - \sin y\right)dx - \int 0\,dy = C$$

$$y\log x - x\sin y = C$$

Ques (9) #

$$(4xy + 3y^2 - x)\,dx + (x^2 + 2xy)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 4x + 6y \qquad \frac{\partial N}{\partial x} = 2x + 2y$$

The D.E is not exact.

Consider, Rule,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4x + 6y - 2x - 2y}{x^2 + 2xy}$$

$$= \frac{2(x + 2y)}{x(x + 2y)} = \frac{2}{x}$$

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx}$$

$$= e^{2\log x} = e^{\log x^2}$$

$$= x^2$$

Multiply x^2 to the D.E

$$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy$$

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y \quad // \quad \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

The DE is exact now.

It General solution.

$$\int (4x^3y + 3x^2y^2 - x^3)dx + \int 0\, dy = C$$

$$x^4y + x^3y^2 - \frac{x^4}{4} = C$$

Ques ⑤ $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

Sol. $\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy$

The D.E is not exact.

Consider Rule 1.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2y} = \frac{4xy}{-x^2y}$$

$$= -\frac{4}{x}$$

H/W

May-13

Homogeneous D.E

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Ques (6) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$\frac{\partial M}{\partial y} = 2x(y^4e^y + e^y 4y^3) + 6xy^2 + 1$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

The D.E is not exact

Consider Rule 2

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-(8xe^y y^3 + 8xy^2 + 4)}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-4(2xe^y y^3 + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)}$$

$$= \frac{-4}{y}$$

$$I.F. = e^{\int \frac{-4}{y}dy} = e^{-4\int \frac{1}{y}dy}$$

$$I.F. = \frac{1}{y^4}$$

Multiply by $\frac{1}{y^4}$ to the D.E

$$\frac{1}{y^4}\{(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy\}$$

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4} \quad \Big|\Big| \quad \frac{\partial N}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

The D.E is Exact now.

It's General solution is:

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \int 0\,dy = C$$

$$\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C}$$

Exercise 1.3 # Homogeneous D.E:

* May-16 *

Ques $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

which is a Homogeneous D.E.

Sol. $I.F. = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y}$

$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$

$= \frac{1}{x^2y^2}$

Multiply I.F to the D.E

$\frac{1}{x^2y^2}(x^2y - 2xy^2) - \frac{(x^3 - 3x^2y)}{x^2y^2}dy$

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \quad // \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$

The D.E is Exact now.

Its General solution:

$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int\frac{3}{y}dy = C$

$\frac{1}{y}x - 2\log x + 3\log y = C$

$\frac{x}{y} + \log y^3 - \log x^2 = C$

$$\boxed{\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C}$$

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Ques 2 $x^2y\,dx - (x^3+y^3)\,dy = 0$

which is H.D.E

Sol $I.F. = \frac{1}{Mx+Ny} = \frac{1}{(x^2y)x-(x^3+y^3)y}$

$= \frac{1}{x^3y - x^3y - y^4}$

$= \frac{1}{-y^4}$

Multiply $\frac{1}{-y^4}$ to the D.E

$-\frac{1}{y^4}(x^2y)\,dx - (x^3+y^3)\times\frac{-1}{y^4}\,dy = 0$

$\left(\frac{-x^2}{y^3}\right)dx + \left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy = 0$

$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$ // $\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

The D.E is exact now.

It's general Solution

$\int \frac{-x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = C$

$\frac{-x^3}{3y^3} + \log y = C$

H/W

Ques 3. $(x^2+xy)\,dy = (x^2+y^2)\,dx$

$(x^2+y^2)\,dx - (x^2+xy)\,dy = 0$

which is H.D.E

Sol $I.F. = \frac{1}{Mx+Ny} = \frac{1}{x^3+\cancel{xy^2} - x^2y - \cancel{xy^2}}$

$= \frac{1}{x^3 - x^2y} = \frac{1}{x^2(x-y)}$

Rule:4 $\quad f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$$I.F. = \frac{1}{Mx - Ny}$$

Q-15
Ques

$$(1-xy)\,y\,dx - (1+xy)\,x\,dy = 0$$

which is of the above type.

$$I.F. = \frac{1}{Mx-Ny} = \frac{1}{(1-xy)yx + (1+xy)xy}$$

$$I.F. = \frac{1}{xy - \cancel{x^2y^2} + xy + \cancel{x^2y^2}} = \frac{1}{2xy}$$

Multiply D.E by $\frac{1}{2xy}$

$$\frac{(1-xy)\cancel{y}}{2x\cancel{y}}\,dx - \frac{(1+xy)\cancel{x}}{2\cancel{x}y}\,dy = 0$$

$$\left(\frac{1-xy}{x}\right)dx - \left(\frac{1+xy}{y}\right)dy = 0$$

$$\left(\frac{1}{x} - y\right)dx - \left(\frac{1}{y} + x\right)dy = 0$$

$$\frac{\partial M}{\partial y} = -1 \quad \Big\Vert \quad \frac{\partial N}{\partial x} = -1$$

Its General solution:-

$$\int\left(\frac{1}{x} - y\right)dx - \int\frac{1}{y}\,dy = C$$

$$\log x - yx - \log y = C$$

$$\boxed{\log\left(\frac{x}{y}\right) - xy = C}$$

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D-13 Ques. $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy=0$

which is of the type 4.

Sol.

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{x^4y^4+x^3y^3+x^2y^2+xy-x^4y^4-x^3y^3-x^2y^2+xy}$$

$$I.F. = \frac{1}{2x^2y^2+2x^3y^3}$$

Multiply D.E by $\frac{1}{2x^2y^2+2x^3y^3}$

$$\frac{1}{2x^2y^2(xy+1)}(x^3y^3+x^2y^2+xy+1)y\,dx + \frac{(x^3y^3-x^2y^2-xy+1)x\,dy}{2x^2y^2(xy+1)}=0$$

$$\frac{x^2y^2(xy+1)+(xy+1)}{2x^2y(xy+1)} + \frac{x^2y^2(xy-1)-(xy-1)}{2xy^2(xy+1)}=0$$

$$\frac{(x^2y^2+1)(xy+1)\,dx}{2x^2y(xy+1)}+\frac{(x^2y^2-1)(xy-1)\,dy}{2xy^2(xy+1)}=0$$

$$\frac{x^2y^2+1}{2x^2y}dx+\frac{(xy-1)(xy-1)(xy+1)\,dy}{2xy^2(xy+1)}=0$$

$$\left(y+\frac{1}{x^2y}\right)dx+\frac{(x^2y^2-2xy+1)\,dy}{xy^2}=0$$

$$\left(y+\frac{1}{x^2y}\right)dx+\left(x-\frac{2}{y}+\frac{1}{xy^2}\right)dy=0$$

The D.E is exact now.

Its General solution is:

$$\int\left(y+\frac{1}{x^2y}\right)dx+\int-\frac{2}{y}\,dy=C$$

$$\cancel{yx - \log}\quad yx-\frac{1}{xy}-2\log y=C$$

$$\left[xy-\frac{1}{xy}-2\log y=C\right]$$

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Ques 5 $(xy\sin xy + \cos xy)y\,dx + (xy\sin yx - \cos xy)x\,dy = 0$

which is of the form

$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

$$I.F. = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy}$$

$$I.F. = \frac{1}{2xy\cos xy}$$

Multiply the D.E by $\frac{1}{2xy\cos xy}$

$$\frac{(xy\sin xy + \cos xy)y\,dx}{2xy\cos xy} + \frac{(xy\sin xy - \cos xy)x\,dy}{2xy\cos xy} = 0$$

$$\left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$$

The D.E is exact now.

The General solution is:

$$\int\left(y\tan xy + \frac{1}{x}\right)dx + \int -\frac{1}{y}\,dy = C$$

$$\frac{y\log\sec(xy)}{y} + \log x - \log y = C$$

$$\# \boxed{\frac{y\log\sec(xy)}{y} + \log\left(\frac{x}{y}\right) = C} \#$$

$$\frac{\partial M}{\partial y} = y\sec^2(xy)\cdot x + \tan xy$$

$$\frac{\partial N}{\partial x} = y\sec^2(xy)\cdot x + \tan xy$$

$$\# \boxed{\log\sec xy + \log\left(\frac{x}{y}\right) = \log C} \#$$

$$\log\left[\frac{\sec xy\cdot x}{y}\right] = \log C$$

Taking Antilog on both sides,

$$\frac{\sec xy \cdot x}{y} = C$$

$$\boxed{\sec(xy) \cdot x = yC}$$

Linear Differential Equation:

Q-15

Ques 1 $\frac{dy}{dx} + \left(\frac{4x}{(x^2+1)}\right) y = \frac{1}{(x^2+1)^3}$

which is of the form

$\frac{dy}{dx} + Py = Q$

where, $P = \frac{4x}{(x^2+1)}$, $Q = \frac{1}{(x^2+1)^3}$

* The General Solution:

① $I.F. = e^{\int P dx}$

$I.F. = e^{\int \frac{4x}{x^2+1} dx} = e^{2\int \frac{2x}{x^2+1} dx}$

$= e^{2\log x^2+1} = e^{\log (x^2+1)^2}$

$I.F. = (x^2+1)^2$

The General Solution:

$y(I.F.) = \int Q.(I.F.) \, dx + C$

$y(x^2+1)^2 = \int \frac{(x^2+1)^2}{(x^2+1)^3} dx + C$

$y(x^2+1)^2 = \int \frac{1}{x^2+1} dx + C$

$\boxed{y(x^2+1)^2 = \tan^{-1}(x) + C}$

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#Linear D.E

M-16

Ques 2. $(1+y^2)dx + (x - e^{\tan^{-1}y})dy = 0$

Linear in x.

which is of the form

$$\frac{dx}{dy} + Px = Q$$

$$(1+y^2)dx = -(x - e^{\tan^{-1}y})dy$$

$$(1+y^2)\frac{dx}{dy} = -(x - e^{\tan^{-1}y})$$

$$1+y^2\frac{dx}{dy} + x = e^{\tan^{-1}y}$$

Divide throughout by $1+y^2$,

$$\therefore \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

where, $P = \frac{1}{1+y^2}$, $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

It is of the form

$$\frac{dx}{dy} + Px = Q.$$

Now, $\underline{I.F} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

The General Solution:

$$x(I.F) = \int Q(I.F)\,dy + C$$

$$x \cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{(1+y^2)} e^{\tan^{-1}y} + C$$

$$x \cdot e^{\tan^{-1}y} = \frac{(e^{\tan^{-1}y})^2}{2} + C$$

N.T.S.C

$\left\{ \int (f(x))^n \cdot f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} \right\}$

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#N·T·S·C = Note this step Carefully#

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Ques 3. $\frac{dy}{dx}\cosh x = 2\cosh^2 \sinh x - y\sinh x$

linear in y.

Sol. $\frac{dy}{dx}\cosh x + y\sinh x = 2\cosh^2 x \sinh x$

Divide by $\cosh x$

$\frac{dy}{dx} + y\tanh x = 2\cosh x \sinh x$

This is of the form

$\frac{dy}{dx} + Py = Q$

where $P = \tanh x$, $Q = 2\cosh x \sinh x$

$I.F. = e^{\int \tanh x dx} = \cancel{e^{\cosh x}} \frac{1}{\sec h x}$

$= \cosh x.$

The General solution:

$y(I.F.) = \int Q \cdot (I.F.)\, dx + C$

$y \cdot e^{\cosh x} = \int 2\cosh x \sinh x \cdot e^{\cosh x} + C$

Put $\cosh x = t$

$\sinh x = dt$

$2\int \cosh x \sinh x = \int 2t^2 = \frac{2t^3}{3}$

$y\cosh x\ \cancel{y \cdot e^{\cosh x}} = \frac{2(\cosh x)^3}{3} + C$

$\# \left[y\cosh x = \frac{2(\cosh x)^3}{3} + C. \right]$ Ans

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Ques. 4 $(x + 2y^3)\frac{dy}{dx} = y \Rightarrow x + 2y^3 = y\frac{dx}{dy}$

(linear in x)

Divide it by y

$$\frac{dx}{dy} - \frac{x + \cancel{2y^3}}{y} = \frac{2y^3}{y} = 2y^2$$

It is of the form

$$\frac{dx}{dy} + Px = Q$$

where $P = \frac{-1}{y}$ and $Q = 2y^2$

$$I.F. = e^{\int \frac{-1}{y} dy} = e^{-\log y}$$

$$= e^{\log y^{-1}}$$

$$= \frac{1}{y}$$

The General Solution:

$$x(I.F) = \int Q\, I.F\, dy + C$$

$$\frac{x}{y} = \int \frac{2y^2}{y} + C$$

$$\left[\frac{x}{y} = y^2 + C\right] \text{ Ans}$$

H/W

Ques. 5 $y\log y\, dx + (x - \log y) dy = 0$

$$\frac{\partial M}{\partial y} = \log y + 1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1 - \log y - 1}{y\log y}$$

$$f(y) = \frac{-1}{y}$$

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$$I.F. = e^{\int P\,dy} = e^{-\int 1/y\,dy}$$
$$= \frac{1}{y}$$

Multiply D.E by $\frac{1}{y}$ we get,

$$\log y\,dx + \left(\frac{x}{y} - \frac{\log y}{y}\right)dy = 0$$

Now, $\frac{\partial M}{\partial y} = \frac{1}{y}$, $\frac{\partial N}{\partial x} = \frac{1}{y}$

The General Solution:

$$\left[2x\log y + (\log y)^2 = C\right] \text{ Ans}$$

Ques. 6 $(y+1)\,dx + \left[x - (y+2)e^y\right]dy = 0$

$$(y+1)\,dx = -\left[x - (y+2)e^y\right]dy$$
$$(y+1)\frac{dx}{dy} = -\left[x - (y+2)e^y\right]$$

Divide it by $(y+1)$

$$\frac{dx}{dy} + \frac{x}{y+1} = \frac{+(y+2)e^y}{y+1}$$

It is of the form $\frac{dx}{dy} + Px = Q$

where $P = \frac{1}{y+1}$, $Q = \frac{(y+2)\,e^y}{y+1}$

$$I.F. = e^{\int \frac{1}{y+1}dy} = e^{\log(y+1)} = y+1.$$

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The General Solution:

$$x \cdot I.F = \int Q \cdot I.F \, dy + C$$

$$x(y+1) = \int \frac{(y+2)e^y}{(y+1)}(y+1)\, dy + C$$

$$x(y+1) = \int (y+2)e^y \, dy + C$$

$$= (y+2)e^y - \int e^y + C$$

$$x(y+1) = (y+2)e^y - e^y + C$$

$$x(y+1) = ye^y + e^y + C$$

$$\# \left[(y+1)(x - e^y) = C\right] \#$$

Ques 7. $(x^2-1)\sin x \frac{dy}{dx} + [2x\sin x + (x^2-1)\cos x]y = (x^2-1)\cos x$

Linear in y.

Divide it by $(x^2-1)\sin x$

$$\frac{dy}{dx} + \frac{2xy}{(x^2-1)} + y\cot x = \cot x$$

$$\frac{dy}{dx} + \frac{2xy}{x^2-1} = \cot x - y\cot x$$

$$\frac{dy}{dx} + \frac{2xy}{x^2-1} = \cot x(1-y)$$

$$\frac{dy}{dx} + \left[\frac{2x}{x^2-1} + \cot x\right]y = \cot x$$

where $P = \frac{2x}{x^2-1} + \cot x$, $Q = \cot x$.

$$I.F = e^{\int\left(\frac{2x}{x^2-1} + \cot x\right)dy} = e^{\log(x^2-1) + \log\sin x}$$

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$$I.F. = e^{\log (x^2-1)\sin x}$$

$$I.F. = (x^2-1)\sin x$$

The General solution.

$$y(I.F.) = \int Q \cdot I.F \, dx + C$$

$$y\,(x^2-1)\sin x = \int (\cot x\,(x^2-1)\sin x)\,dx + C$$

$$= \int \left(\frac{\cos x}{\sin x}\,(x^2-1)\sin x\right)dx + C$$

$$= \int (x^2-1)\cos x \, dx + C$$

$$y(x^2-1)\sin x = (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + C$$

$$y\,(x^2-1)\sin x = (x^2-1)\sin x + 2x\cos x - 2\sin x + C$$

$$(x^2-1)\sin x\,(y-1) = 2x\cos x - 2\sin x + C$$

H.W

Ques ② $\frac{dy}{dx}(\tan x) + y = \cos^2 x \sin x$

$$\frac{dy}{dx} + \frac{y}{\tan x} = \cos^2 x \cdot \frac{\cos x}{\cancel{\sin x}} \cdot \cancel{\sin x}$$

$$\frac{dy}{dx} + y\cot x = \cos^3 x$$

Here, $P = \cot x$, $Q = \cos^3 x$

$$\therefore \quad I.F = e^{\int P dx} = e^{\int \cot x\, dx}$$
$$= e^{\log \sin x}$$
$$= \sin x$$

The General Solution:

$$y(I.F) = \int Q(I.F)\,dx + C$$

$$y\sin x = \int \cos^3 x \sin x\, dx + C$$

$\cos x = t$

$\therefore \sin x\, dx = -dt$

$$\int \cos^3 x \cdot \sin x\, dx = -\int t^3 dt$$
$$= -t^4/4$$

$$\boxed{y\sin x = \frac{-\cos^4 x}{4} + C} \text{ Ans}$$

Ques ③ $(1+x+xy^3)dy + (y+y^2)dx = 0$

$$\frac{dM}{dy} = (1+3y^2) \;,\; \frac{dN}{dx} = 1+y^2$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1+y^2-1-3y^2}{y+y^3}$$

$$= \frac{-2y^2}{y(1+y^2)}$$

$$f(y) = \frac{-2y}{1+y^2}$$

$$I.F. = e^{\int f(y)\,dy} = e^{-\int \frac{2y}{1+y^2}\,dy}$$

$$= \frac{1}{1+y^2}$$

Multiply D.E by I.F we get

$$\frac{(1+x+xy^2)}{1+y^2}\,dy + y\,dx = 0$$

$$\left(\frac{1}{1+y^2} + x\right)dy + y\,dx = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

D.E is exact now.

The General Solution:

$$\int M\,dx + \int (\text{terms in } N \text{ free from } x) = C$$

$$xy + \tan^{-1} y = C$$

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Ans 9 $(y-2x^3)dx - x(1-xy)dy = 0$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1+2xy$

$\therefore$ D.E is not exact

Consider, Rule 1,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1+-2xy}{-x(1-xy)} = \frac{-2}{x}$$

$$I.F. = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x}$$

$$= \frac{1}{x^2}$$

Multiply D.E by $\frac{1}{x^2}$

$$\left(\frac{y}{x^2} - 2x\right)dx - \left(\frac{1}{x} - y\right)dy = 0$$

$\frac{\partial M}{\partial y} = \frac{1}{x^2}$ || $\frac{\partial N}{\partial x} = \frac{1}{x^2}$

D.E is exact now.

$\therefore$ The G.S is

$$\int M dx + \int \left[\begin{matrix}\text{terms in } N \text{ free} \\ \text{from } x\end{matrix}\right] dy = C$$

$$\int \left(\frac{y}{x^2} - 2x\right)dx + \int y\,dy = C$$

$$\frac{-y}{x} - x^2 + \frac{y^2}{2} = C$$

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Ques 5 $(2x^3y - x^2\cos y)dy = (\tan y - 3x^4)dx$

$$\frac{\partial M}{\partial y} = \sec^2 y \quad , \frac{\partial N}{\partial x} = -\sec^2 y + 2x\cos y$$

D.E is not exact

Consider Rule 1,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2(\sec^2 y - x\cos y)}{x(\sec^2 y - x\cos y)} = \frac{-2}{x}$$

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2}$$

Multiplying given D.E by $\frac{1}{x^2}$

$$\left(\frac{\tan y}{x^2} - 3x^2\right)dx - \left(\frac{\sec^2 y}{x} - \cos y\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\sec^2 y}{x^2} \quad // \quad \frac{\partial N}{\partial x} = \frac{\sec^2 y}{x^2}$$

$\therefore$ DE is exact now.

The General solution:

$$\int M dx + \int \text{terms in } N \text{ free from } x = C$$

$$\int\left(\frac{\tan y}{x^2} - 3x^2\right)dx + \int \cos y \, dy = C$$

$$-\frac{\tan y}{x} - x^3 + \sin y = C$$

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Ques 6 $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$

$$\frac{\partial M}{\partial y} = 12x^2y^3+2x \quad , \frac{\partial N}{\partial x} = 6x^2y^3-2x$$

Consider Rule 2,

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{-2(3x^2y^3+2x)}{y(3x^2y^3+2x)} = \frac{-2}{y}$$

$$I.F. = e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy}$$

$$= \frac{1}{y^2}$$

Multiplying D.E by $\frac{1}{y^2}$

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

$$\frac{\partial M}{\partial y} = 6x^2y-\frac{2x}{y^2} \quad , \frac{\partial N}{\partial x} = 6x^2y-\frac{2x}{y^2}$$

D.E is exact now.

The General Solution

$$\int M dx + \int (\text{terms in } N \text{ free from } x) dy = C$$

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx+\int 0\,dy=C$$

$$x^3y^2+\frac{x^2}{y}=C$$

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Ans 7. $(x^2+xy)\,dy = (x^2+y^2)\,dx$

$$(x^2+y^2)dx - (x^2+xy)\,dy = 0$$

It is Homogeneous D.E

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{(x^2+y^2)x-(x^2+yx)y}$$

$$= \frac{1}{x^2(x-y)}$$

Multiplying DE by $\frac{1}{x^2(x-y)}$

$$\left(\frac{1}{x-y} + \frac{y}{x(x-y)}\right)dy + \left(\frac{1}{x-y} + \frac{y^2}{x^2(x-y)}\right)dx = 0$$

$$\frac{dM}{dy} = \frac{1}{(x-y)^2} + \frac{2y(x-y)}{x^2(x-y^2)^2} + \frac{y^2}{x^2(x-y^2)}$$

$$= \frac{1}{(x-y)^2} + \frac{y(2x-y)}{x^2(x-y)^2}$$

$$\frac{dN}{dx} = \frac{1}{(x-y)^2} + \frac{y(2x-y)}{x^2(x-y)^2}$$

D.E is exact now.

The General Solution:

$$\int\left(\frac{1}{(x-y)} + \frac{y^2}{x^3-x^2y}\right)dx + \int 0\,dy = C$$

$$\frac{2\log(x-y)}{x^2-2xy} + 12y^2\frac{\log(x^3-x^2y)}{3x^4-4x^3y} = C$$

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Reducible to LDE

Ques 8 $3y^2 \frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$

$\frac{dy}{dx}$; $\frac{f(y) \text{ and } f'(y)}{P \text{ and } Q \text{ f}^n \text{ of } x}$

Put $y^3 = V$

Diff wrt 'x'

$3y^2 \frac{dy}{dx} = \frac{dV}{dx}$

The D.E becomes,

$\frac{dV}{dx} + 2xV = 4xe^{-x^2}$

It is Linear in V

where $P = 2x$, $Q = 4xe^{-x^2}$

$I.F. = e^{\int P dx} = e^{\int 2x} = e^{\frac{2x^2}{2}}$

$= e^{x^2}$

The General Solution:

$V(I.F) = \int Q(I.F)dx + C$

$Ve^{x^2} = \int 4x \cancel{e^{-x^2}} \cancel{e^{x^2}} dx + C$

$Ve^{x^2} = 2x^2 + C$

Putting $V = y^3$

$[y^3 e^{x^2} = 2x^2 + C]$ Ans

Ques 4 $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

Divide throughout by $\sqrt{y}$

$\frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{x}{(1-x^2)} \sqrt{y} = x$

Put $\sqrt{y} = v$

Diff wrt 'x'

$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} = \frac{2dv}{dx}$

The D.E ~~eq~~ becomes

$\frac{2dv}{dx} + \frac{x}{(1-x^2)} v = x \Rightarrow \frac{dv}{dx} + \frac{xv}{2(1-x^2)} = \frac{x}{2}$

It is linear in v

where $P = \frac{x}{2(1-x^2)}$, $Q = \frac{x}{2}$

$I.F = e^{\int P dx} = e^{\int \frac{x}{2(1-x^2)} dx}$

$= e^{\frac{1}{2}\int \frac{x}{1-x^2} dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2} dx}$

$= e^{\frac{1}{-4} \log(1-x^2)}$

$(I.F.) = (1-x^2)^{-1/4}$

The General Solution:

$v(I.F.) = \int Q(I.F) dx + C$

$v(1-x^2)^{\frac{-1}{4}} = \int \frac{x}{2} (1-x^2)^{\frac{-1}{4}} dx + C$

$= \frac{-1}{4} \int (1-x^2)^{-1/4} (-2x) dx + C$

$v(1-x^2)^{-1/4} = \frac{-1}{4} \frac{(1-x^2)^{3/4}}{3/4} + C$

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Resub $v = \sqrt{y}$

$$\left[\sqrt{y}\,(1-x^2)^{-\frac{1}{4}} = \frac{-(1-x^2)^{3/4}}{3} + C\right] \text{ Ans}$$

Ques 9.

$$\frac{dy}{dx} + x^3\sin^2 y + x\sin 2y = x^3$$

$$\frac{dy}{dx} + x\sin 2y = x^3(1-\sin^2 y)$$

$$\frac{dy}{dx} + x\, 2\sin y\cos y = x\cos^2 y$$

Divide by $\cos^2 y$ throughout

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = v$

Diff w.r.t 'x'

$$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$$

The DE becomes

$$\frac{dv}{dx} + 2xv = x^3$$

It is linear in v

where, $P = 2x$, $Q = x^3$

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

The General Solution:

$$v(I.F) = \int Q(I.F)\,dx + C$$

$$ve^{x^2} = \int x^3 e^{x^2}\,dx + C$$

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Put

$$t = x^2$$

$$2x\,dx = dt$$

$$\frac{dt}{2} = x\,dx$$

$$ve^{x^2} = \int x^2 \cdot e^{x^2} x\,dx + C$$

$$= \int t \cdot e^t \frac{dt}{2} + C$$

$$= \frac{1}{2}\left[te^t - 1e^t\right] + C$$

$$ve^{x^2} = \frac{1}{2}\left[te^t - 1e^t\right] + C$$

Resub $t = x^2$

$$ve^{x^2} = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + C$$

Again subs $v = \tan y$

$$e^{x^2}\tan y = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + C$$

$$2e^{x^2}\tan y = e^{x^2}(x^2-1) + C$$

$$2\tan y = (x^2-1) + C$$

Ques 10 $$\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$$

Divide by $\frac{1}{1+y^2}$ throughout

$$\frac{1}{1+y^2}\frac{dy}{dx} + (2x\tan^{-1}y - x^3) = 0$$

$$\frac{1}{1+y^2}\frac{dy}{dx} + 2x\tan^{-1}y = x^3$$

Put $\tan^{-1}y = v$

Diff wrt 'x'

$$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + 2xv = x^3$$

It is Linear in v

where $P = 2x$, $Q = x^3$

$$I.F. = e^{x^2}$$

The General Solution:

$$v(I.F.) = \int Q(I.F.)\,dx + C$$

$$v e^{x^2} = \int x^3 e^{x^2}\,dx + C$$

$$= x^2 e^{x^2} \cdot x\,dx + C$$

Put $x^2 = t$

$$x\,dx = \frac{dt}{2}$$

$$ve^{x^2} = \int t\, e^t \frac{dt}{2} + C$$

$$ve^{x^2} = \frac{1}{2}(te^t - e^t) + C$$

Put $t = x^2$ and $v = \tan^{-1}y$

$$\tan^{-1}y\, e^{x^2} = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + C$$

$$2\tan^{-1}y\, e^{x^2} = e^{x^2}(x^2 - 1) + C$$

$$2\tan^{-1}y = (x^2 - 1) + C$$

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Ques 3 (1) $\frac{dy}{dx} = x^3y^3 - xy$ {Alway Keep higher power terms on RHS}

Sol.

$\frac{dy}{dx} + xy = x^3y^3$

Divide throughout by y^3

$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$

Put $v = \frac{1}{y^2}$

Diff wrt 'x'

$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{1}{y^3}\frac{dy}{dx} = \frac{-dv}{2dx}$

$\frac{-dv}{2dx} + xv = x^3$

$\frac{dv}{dx} + (2xv) = -2x^3$

$\frac{dv}{dx} - 2xv = -2x^3$

It is linear in v

where $P = -2x$, $Q = -2x^3$

$I.F = e^{-2\int x dx} = e^{-x^2}$

The General Solution:

$v(I.F) = \int Q(I.F)dx + C$

$ve^{-x^2} = \int -2x^3 . e^{-x^2}dx + C$

$= \int 2x^2 e^{-x^2} \cdot x dx + C$

Put $-x^2 = t$

$\therefore xdx = \frac{-dt}{}$

$$ve^{-x^2} = -2\int t\, e^t \frac{dt}{2} + C$$

$$= -\int t\, e^t\, dt + C$$

$$ve^{-x^2} = -[te^t - e^t] + C$$

Putting again $-x^2 = t$

$$\therefore\ ve^{-x^2} = -[-x^2e^{-x^2} - e^{-x^2}] + C$$
$$= x^2e^{-x^2} + e^{-x^2} + C$$

$$ve^{-x^2} = e^{-x^2}(x^2+1) + C$$

$$\#[v = (x^2+1) + C]\#$$

Resub $v = \frac{1}{y^2}$

$$\left[\frac{1}{y^2} = (x^2+1) + C\right] \text{Ans}$$

(ii) $(x^3y^3 - xy)dy = dx$

Sol. $x^3y^3 - xy = \frac{dx}{dy}$

$$\frac{dx}{dy} + xy = x^3y^3$$

Divide throughout by x^3

$$\frac{1}{x^3}\frac{dx}{dy} + \frac{y}{x^2} = y^3$$

Put $\frac{1}{x^2} = v$

Diff wrt y

$$\frac{-2}{x^3}\frac{dx}{dy} = \frac{dv}{dy}$$

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$$\frac{1}{3}\frac{dx}{dy} = \frac{-1}{2}\frac{dv}{dy}$$

$$\frac{-1}{2}\frac{dv}{dy} + vy = y^3$$

$$\frac{dv}{dy} - 2vy = -2y^3$$

It is linear in v

where, $P = -2y$, $Q = -2y^3$

$$I.F. = e^{\int P dy} = e^{-2\int y\,dy}$$
$$= e^{-y^2}$$

The General Solution is:

$$v(I.F.) = \int Q(I.F.)dy + C$$

$$ve^{-y^2} = \int -2y^3 \cdot e^{-y^2} dy + C$$

Putting $-y^2 = t$

$$y\,dy = -\frac{dt}{2}$$

$$\therefore \quad ve^{t} = 2\int te^{t}\frac{dt}{2} + C$$

$$ve^{t} = -\int te^{t}dt + C$$

$$ve^{t} = -[te^{t} - e^{t}] + C$$

Putting $v = \frac{1}{x^2}$ and $t = -y^2$

$$\# \left[\frac{1}{x^2}e^{-y^2} = e^{-y^2}(y^2+1) + C\right] \#$$ Ans

$$\# \left[\frac{1}{x^2} = (y^2+1) + C\right] \#$$ Ans

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Ques 5 $y\,dx + x(1-3x^2y^2)\,dy = 0$

Sol. $y\,dx = -x(1-3x^2y^2)\,dy$

$$y\frac{dx}{dy} = -x(1-3x^2y^2)$$

$$y\frac{dx}{dy} = -x + 3x^3y^2)$$

$$y\frac{dx}{dy} + x = 3x^3y^2$$

Divide throughout by x^3

$$\frac{y\,dx}{x^3dy} + \frac{1}{x^2} = 3y^2$$

$$\frac{dx}{x^3dy} + \frac{1}{x^2y} = 3y$$

Put $\frac{1}{x^2} = v$

Diff w.r.t 'y'

$$\frac{-2}{x^3}\frac{dx}{dy} = \frac{dv}{dy}$$

$$\frac{1}{x^3}\frac{dx}{dy} = \frac{-1}{2}\frac{dv}{dy}$$

The DE becomes

$$\frac{-1}{2}\frac{dv}{dy} + \frac{v}{y} = 3y$$

Multiply by -2 on both sides,

$$\frac{dv}{dy} - \frac{2v}{y} = -6y$$

Here $P = -\frac{2}{y}$ $\quad Q = -6y$

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$$I.F. = e^{\int P\,dy} = e^{\int -\frac{2}{y}\,dy}$$

$$I.F. = \frac{1}{y^2}$$

The General Solution:

$$v(I.F.) = \int Q(I.F.)\,dy + C$$

$$v \cdot \frac{1}{y^2} = \int -6y \cdot \frac{1}{y^2}\,dy + C$$

$$\frac{v}{y^2} = \int -\frac{6}{y}\,dy + C$$

$$\frac{v}{y^2} = -6\log y + C$$

Re Subs $v = \frac{1}{x^2}$

$$\left[\frac{1}{x^2y^2} = -6\log y + C\right]$$ Ans

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Que 6 $y^4 dx = (x^{-3/4} - y^3 x) dy$

Sol:

$$y^4 \frac{dx}{dy} = x^{-3/4} - y^3 x$$

$$y^4 \frac{dx}{dy} + y^3 x = x^{-3/4}$$

Divide throughout by $x^{-3/4} y^4$

$$x^{3/4} \frac{dx}{dy} + \frac{x^{7/4}}{y} = \frac{1}{y^4}$$

Put $x^{7/4} = v$

Diff wrt y,

$$\frac{7}{4} x^{3/4} \frac{dx}{dy} = \frac{dv}{dy}$$

$$x^{3/4} \frac{dx}{dy} = \frac{4}{7} \frac{dv}{dy}$$

$$\frac{4}{7} \frac{dv}{dy} + \frac{v}{y} = \frac{1}{y^4}$$

$$\frac{dv}{dy} + \frac{7v}{4y} = \frac{7}{4y^4}$$

$P = \frac{7}{4y}$, $Q = \frac{7}{4y^4}$

$$I.F = e^{\int P dy}$$

$$= e^{7/4 \int \frac{1}{y} dy}$$

$$= e^{7/4 \log y}$$

$$= e^{\log y^{7/4}}$$

$$I.F = y^{7/4}$$

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The General Solution:

$$v \cdot (I \cdot F) = \int Q(I \cdot F)\, dy + C$$

$$v y^{7/4} = \frac{7}{4} \int \frac{1}{y^4}\, y^{7/4}\, dy + C$$

$$x^{7/4} y^{7/4} = \frac{7}{4}\, y^{-9/4} + C$$

$\left[x^{7/4} \cdot y^{7/4} = \frac{7}{4} y^{-9/4} + C \right]$ Ans

Date 28/01/19

Chapter-2

Linear Differential Equation with Constant / Variable Coefficient Of Higher Order

Consider,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{2x}$$

$$\left(\frac{d^2}{dx^2} + \frac{5d}{dx} + 6\right)y = e^{2x}$$

Let $\frac{d}{dx} = D$

$$\frac{d^2}{dx^2} = D^2$$

$$\vdots \qquad \vdots$$

$$(D^2 + 5D + 6)y = e^{2x}$$

i.e. $f(D)y = Q$

The solution of the DE :-

Complete solⁿ = Complementary function + Particular function Integral

$$\boxed{y = y_c + y_p}$$

* Step ①: To find C.F (y_c):

$$f(D) = 0$$

$$D^2 + 5D + 6 = 0$$

$$D = -2, -3.$$

Nature of roots	Example	C.F.
Real and Distinct	$-2, -3$	$y_c = C_1e^{-2x} + C_2e^{-3x}$
Real & Repeated	$-3, -3$	$y_c = (C_1 + C_2x)e^{-3x}$
	$4, 4, 4$	$y_c = (C_1 + C_2x + C_3x^2)e^{4x}$
Complex & Distinct	$2 \pm 3i$	$y_c = e^{2x}[C_1\cos 3x + C_2 \sin 3x]$
Complex & ~~Distinct~~ Repeated	$2 \pm 3i$, $2 \pm 3i$	$y_c = e^{2x}[(C_1 + C_2x)\cos 3x + (C_3 + C_4x)\sin 3x]$
Repeated & Imaginary	$\pm i, \pm i$	$y_c = e^{0x}[(C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x]$

Tricky Sums to Find C.F.:

$$D^4 + a^2 = 0$$

Add and subtracting "$2D^2a$"

$$D^4 + 2D^2a + a^2 - 2D^2a = 0$$

$$(D^2 + a)^2 - (\sqrt{2}Da)^2 = 0$$

To Find Particular Integral:

Case ①: $Q = e^{ax} + b$ // $Q = e^{ax+b}$

In general, $y_p = \frac{1}{F(D)} Q$

$$= \frac{1}{F(D)} (e^{ax+b})$$

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Replace $D = a$

$$y_p = \frac{1}{F(a)} e^{ax+b}$$

Ex:

$$\frac{1}{D^2+5D+6} (e^{2x})$$

$$\frac{1}{20} e^{2x}$$

But if $f(a) = 0$

$$y_p = \frac{1}{f(D)} (e^{ax+b})$$

$$y_p = \frac{x \cdot 1}{f'(D)} (e^{ax+b})$$

Replace $D = a$

$$y_p = \frac{x}{f'(a)} (e^{ax+b})$$

Suppose again $f'(a) = 0$

$$y_p = \frac{x^2}{f''(D)} (e^{ax+b})$$

Case-II $\quad Q = \sin(ax+b)$ or $\cos(ax+b)$

Replace, $D^2 = -a^2$

$D^3 = -a^2 D$

$D^4 = a^4$

$D^5 = a^4 D$

To Solve D-terms:-

① $\frac{1}{D+a} = \frac{D-a}{D^2-a^2}[\sin(ax)] = \frac{\left[\frac{d}{dx} - a\right]\sin(ax)}{-a^2-a^2}$ $\{D^2 = -a^2\}$

② $\frac{1}{D}\sin(ax) = \int \sin(ax)\,dx$

#Note: In Q use the following idea to bring into standard form:

① $K = Ke^{0x}$

② $\sin^2 x = \frac{1-\cos 2x}{2}$

③ $\cos^2 x = \frac{1+\cos 2x}{2}$

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Case-III :- $\underline{Q = x^n}$

$$y_p = \frac{1}{F(D)} \cdot x^n$$

[① Take least power term common from the denominator]

$$= \frac{1}{[1 + G(D)]} \cdot x^n$$

② Now take $[1 + G(D)]$ in the Numerator.

$$\therefore y_p = [1 + G(D)]^{-1} x^n$$

$$y_p = [1 - D + D^2 - D^3] x^n$$

③ Expand $[1 + G(D)]^{-1}$ using the following formulae.

$$y_p = x^n - nx^{n-1} + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 - \cdots$$

Ex:

$$y_p = \frac{1}{D^2 + 5D + 6} x^2 = \frac{1}{6\left[1 + \frac{5D}{6} + \frac{D^2}{6}\right]} x^2$$

ascending order.

$$= \frac{1}{6}\left[1 + \underbrace{\left(\frac{5D}{6} + \frac{D^2}{6}\right)}_{\text{In bracket}}\right]^{-1} x^2$$

$$= \frac{1}{6}\left[1 - \left(\frac{5D}{6} + \frac{D^2}{6}\right) + \left(\frac{5D}{6} + \frac{D^2}{6}\right)^2 \cdots\right] x^2$$

Case 5: $Q = x \cdot V$

$$y_p = \frac{1}{f(D)} \cdot xV$$

$$= \left[x - \frac{1}{f(D)} \cdot f'(D)\right] \frac{1}{f(D)} \cdot V$$

Note:

The above formula can be used only if

(i) The powers of D is even

(ii) The power of x is 1.

Case VI:

Q = Any function of x. $\left\{\sin^{-1}x, \log x, \frac{1}{\sqrt{1+x^2}}\right\}$

$$y_p = \frac{1}{f(D)} \cdot Q$$

$$y_p = \frac{1}{(D-n)(D-m)} \cdot Q$$

Write $f(D)$ into its factors.
Then apply factors turn by turn.

$$y_p = \frac{1}{(D-n)}\left[\frac{1}{(D-m)} \cdot Q\right]$$

$$\frac{1}{(D-m)} \cdot Q = e^{mx}\int e^{-mx} \cdot Q\,dx$$

$$\frac{1}{D+m} \cdot Q = e^{-mx}\int e^{mx} \cdot Q\,dx$$

$$\frac{1}{D} Q = \int Q \cdot dx$$

Cauchy's L.D.E with ~~Constant~~ Variable Coefficient :-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = Q$$

Put $\boxed{x = e^z}$ $\boxed{\log x = z}$

$$\frac{dx}{dy} = e^z \frac{dz}{dy}$$ [Diff wrt 'y']

$$\frac{dy}{dx} = \frac{1}{e^z}\frac{dy}{dz}$$ {Reciprocal of above fn}

$$x\frac{dy}{dx} = \frac{dy}{dz}$$ {Putting $e^z = x$ Again}

Now, $\left[D = \frac{d}{dz}\right]$

$$\therefore \boxed{\begin{aligned} x\frac{dy}{dx} &= Dy \\ x^2\frac{d^2y}{dx^2} &= D(D-1)y \\ x^3\frac{d^3y}{dx^3} &= D(D-1)(D-2)y \end{aligned}}$$

(Degree of x should be same as order of derivative)

Date 14/02/19

Legendre's L.D.E : It is of the form

$$(ax+b)^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = Q$$

Let $(ax+b) = e^z$

$\log(ax+b) = z$

$(ax+b)\frac{dy}{dx} = aDy$ $\left[D = \frac{d}{dz}\right]$

$(ax+b)^2 \frac{dy}{dx} = a^2 D(D-1)y$

$(ax+b)^3 \frac{dy}{dx} = a^3 D(D-1)(D-2)y$

Methods of Variation of Parameters:

The sum here is of the same type as Case VI i.e. Q = any fn

$F(D)y = Q$

Let us say,

$y_c = C_1 e^x + C_2 e^{-x}$

$y_p = uy_1 + vy_2$

$y_1 = e^x, \; y_2 = e^{-x}$

$u = -\int \frac{y_2 Q}{W} dx$

$v = -\int \frac{y_1 Q}{W} dx$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Case 1: $Q = e^{ax}$

Ques 1 Find the C.F:-

(i) $(D^4 + 8D^2 + 16)y = 0$

$F(D)y = Q$

Sol. To Find C.F: The Auxiliary Equation:

$F(D) = 0$

$D^4 + 8D^2 + 16 = 0$

$(D^2 + 4)^2 = 0$

Consider,

$D^2 + 4 = 0$

$D^2 = -4$

$D = \sqrt{-4}$

$D = \pm 2i, \pm 2i$

$$y_c = e^{0x}\left[(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x\right]$$

(ii) $(D^2 - 1)(D - 1)^2 y = 0$

The Auxiliary Equation

$F(D) = 0$

$(D^2 - 1)(D - 1)^2 = 0$

$D^2 - 1 = 0$ $\quad\quad$ $(D - 1)^2 = 0$

$D = \pm 1$ $\quad\quad$ $D = 1, 1$

$$y_c = (C_1 + C_2 x + C_3 x^3)e^x + C_4 e^{-x}$$

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(iii) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Sol. $(D^3 - 6D^2 + 11D - 6)y = 0$

The Auxiliary Equation;

$D^3 - 6D^2 + 11D - 6 = 0$

By Guess Work.

$D = 1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & \downarrow & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$(D-1)(D^2 - 5D + 6) = 0$

$D = 1, 2, 3$

$[y_c = C_1e^x + C_2e^{2x} + C_3e^{3x}]$ Ans

(iv) $(D^4 - 4)y = 0$

The Auxiliary Equation:

$D^4 - 4 = 0$

$(D^2 - 2)(D^2 + 2) = 0$

$D^2 = 2$, $D^2 = -2$

$D = \pm\sqrt{2}$, $D = \pm\sqrt{2}i$

$y_c = C_1e^{\sqrt{2}x} + C_2e^{-\sqrt{2}x} + e^0[C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}x]$

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(v) $(D^4+4)y = 0$

Sol. The Auxiliary Equation

$F(D) = 0$

$D^4+4 = 0$

$D^4+4D^2+4-4D^2 = 0$

$(D^2+2)^2 - (2D)^2 = 0$

$a+b$	$a-b$
$D^2+2D+2 = 0$	$D^2-2D+2 = 0$
$D = \frac{-2 \pm \sqrt{-4}}{2}$	$D = \frac{2 \pm \sqrt{-4}}{2}$
$D = -1 \pm i$	$D = 1 \pm i$

$$y_c = e^{-x}[C_1\cos x + C_2\sin x] + e^{x}[C_3\cos x + C_4\sin x]$$

(vi) $(D^4+1)y = 0$

Sol. The Auxiliary Equation,

$F(D) = 0$

$D^4+1 = 0$

$\cancel{D^4+1=0}$ $D^4+2D^2+1-2D^2 = 0$

$(D^2+1)^2 - (\sqrt{2}D)^2 = 0$

$a+b$	$a-b$
$D^2+1+\sqrt{2}D = 0$	$D^2-\sqrt{2}D+1 = 0$
$D = \frac{-\sqrt{2} \pm \sqrt{-2}}{2}$	$D = \frac{\sqrt{2} \pm \sqrt{-2}}{2}$
$= \frac{-\sqrt{2} \pm i\sqrt{2}}{2}$	$D = \cancel{\frac{1 \pm i}{2}}\ \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$

$\cancel{y_c = e^{-x/2}[C_1\cos x + C_2\sin x] + e^{x/2}[C_3\cos x + C_4\sin x]}$

$$y_c = e^{-1/\sqrt{2}}[C_1\cos(1/\sqrt{2}) + C_2\sin(1/\sqrt{2})] + e^{1/\sqrt{2}}[C_3\cos x/\sqrt{2} + C_4\sin 1/\sqrt{2}]$$

Particular Integrals

Ques 2. Find Particular Integral:

(7) $(D^3-2D^2-5D+6)y = e^{3x+8}$

$$y_p = \frac{1}{F(D)}(e^{ax+b})$$

$$= \frac{1}{(D^3-2D^2-5D+6)}(e^{3x+8})$$

At ~~Replace~~ $D=a$ [i.e. $D=3$]

$F(D) = 0$

$$y_p = \frac{x}{(3D^2-4D-5)}(e^{3x+8})$$

Replace $D=3$,

$$y_p = x\left[\frac{1}{10}e^{3x+8}\right]$$

$$y_p = \frac{x}{10}(e^{3x+8})$$

$F(D)=0$ ⇒ $(D^3-2D^2-5D+6)y = 0$ ⇐ The Auxiliary Equation

By guess work, $D=1$ is a root.

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$(D-1)(D^2-D-6)=0$

$D = 1, -2, 3$

$$\boxed{y_c = C_1e^{x} + C_2e^{-2x} + C_3e^{3x}}$$

Ques 2. $\left(6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y\right) = e^{-3x/2} + 2^x$

$$\left(6\frac{d^2}{dx^2} + 17\frac{d}{dx} + 12\right)y = e^{\frac{-3x}{2}} + e^{x\log 2}$$

$$b = e^{\log_e b}$$
$$2^x = e^{\log 2^x}$$
$$= e^{x \log 2}$$

$$\frac{d^2}{dx^2} = D^2, \quad \frac{d}{dx} = D$$

$$(6D^2 + 17D + 12)y = 0$$

To find C.F. (y_c):

The Auxiliary Equation,

$$6D^2 + 17D + 12 = 0$$

$$D = \frac{-3}{2}, \frac{-4}{3}$$

$$y_c = C_1 e^{\frac{-3x}{2}} + C_2 e^{\frac{-4x}{3}}$$

* To find P.I: (y_p)

$$y_p = \frac{1}{f(D)} e^{ax+b} = \frac{1}{(6D^2+17D+12)}\left(e^{\frac{-3}{2}x} + e^{x\log 2}\right)$$

$$= \frac{1}{6D^2+17D+12}\left(e^{-3x/2}\right) + \frac{1}{6D^2+17D+12}\left(e^{x\log 2}\right)$$

Putting $D = \frac{-3}{2}$,

Deno. becomes zero

$$y_p = \frac{x}{12D+17} e^{-3x/2} + \frac{e^{x\log 2}}{6(\log 2)^2 + 17\log 2 + 12}$$

$$y_p = \left[x\left(\frac{e^{-3/2}}{-1}\right) + \frac{e^{x\log 2}}{17.661}\right]$$ Ans

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$$\# \left[y_p = -xe^{-3x/2} + \frac{e^{x\log 2}}{6(\log 2)^2 + 17(\log 2) + 12} \right] \text{Ans}$$

The C.S:- $[y = y_c + y_p]$

Ques 3 $(D^3 - 1)y = (e^x + 1)^2$

The Auxiliary Equation:

$$D^3 - 1 = 0$$

$$(D-1)(D^2 + D + 1) = 0$$

$$D = 1, \; D = \frac{-1 \pm \sqrt{3}}{2}$$

$$y_c = C_1 e^x + e^{-x/2}\left[C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$Q = (e^x + 1)^2 = e^{2x} + 2e^x + 1$$

#N.T.S.C $\Rightarrow \left[Q = e^{2x} + 2e^x + e^{0x}\right]$

$$y_p = \frac{1}{D^3 - 1}\left[e^{2x} + 2e^x + e^{0x}\right]$$

$$= \frac{e^{2x}}{D^3 - 1} + 2\frac{1}{D^3 - 1}e^x + \frac{1}{D^3 - 1}e^{0x}$$

$$= \frac{1}{7}e^{2x} + \frac{2x}{3D^2}e^x + \frac{1}{D^3 - 1}e^{0x}$$

$$= \frac{1}{7}e^{2x} + \frac{2xe^x}{3} + \frac{1}{-1}e^{0x}$$

$$= \frac{1}{7}e^{2x} + \frac{2}{3}xe^x - 1e^{0x}$$

$$y_p = \frac{1}{7}e^{2x} + \frac{2}{3}xe^x - 1$$

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Ques 4 $(D^3-4D)y = 2\cosh 2x = 2\dfrac{(e^{2x}+e^{-2x})}{2} = e^{2x}+e^{-2x}$

Sol. The Auxiliary Equation:

$$D^3-4D=0$$
$$D(D^2-4)=0$$
$$D=0, \quad D^2=4$$
$$D=\pm 2$$
$$D=0, 2, -2.$$

$$y_c = C_1 + C_2e^{2x} + C_3e^{-2x}.$$

$$y_p = \frac{1}{D^3-4D} e^{2x}+e^{-2x}$$
$$= \frac{1}{D^3-4D}e^{2x} + \frac{e^{-2x}}{D^3-4D}$$

At 2, $D=0$ At $D=-2$, $D=0$.

$$= \frac{x\, e^{2x}}{3D^2-4} + \frac{x\, e^{-2x}}{3D^2-4}$$
$$= \frac{x}{8}e^{2x} + \frac{x}{8}e^{-2x}$$
$$= \frac{x}{4}\left(\frac{e^{2x}+e^{-2x}}{2}\right)$$
$$y_p = \frac{x}{4}\cosh 2x$$

$\therefore$ $y = y_c + y_p$

$$y = C_1 + C_2e^{2x} + C_3e^{-2x} + \frac{x}{4}\cosh 2x$$

Case-II :-

$Q = \sin(ax+b)$

Ques 1

$(D^2 - 5D + 6)y = \sin 3x$

$D = 2, 3$

$$\boxed{y_c = C_1 e^{2x} + C_2 e^{3x}}$$

$$y_p = \frac{1}{D^2 - 5D + 6}(\sin 3x)$$

$D^2 = -9$

$$y_p = \frac{1}{-9 - 5D + 6}\sin 3x = \frac{1}{-5D - 3}(\sin 3x)$$

$$= -\frac{(5D-3)}{(5D+3)(5D-3)}\sin 3x = \frac{-(5D-3)\sin 3x}{25D^2 - 9}$$

$$= \frac{-[5D(\sin 3x) - 3\sin 3x]}{-234 - 9} \quad \{\because D^2 = -9\}$$

$$= \frac{-[5 \cdot 3\cos 3x - 3\sin 3x]}{-234}$$

$$= \frac{3(5\cos 3x - \sin 3x)}{\cancel{234}\ 78}$$

$$y_p = \frac{5\cos 3x - \sin 3x}{78}$$

Ques (2) $(D^2 + D + 1)y = (1 + \sin x)^2$

Auxiliary equation: $D^2 + D + 1 = 0$

$$D = \frac{-1 \pm i\sqrt{3}}{2}$$

$$y_c = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$Q = (1 + \sin x)^2$$

$$= 1 + 2\sin x + \sin^2 x$$

$$= e^{0x} + 2\sin x + \frac{1 - \cos 2x}{2}$$

$$= 1 + 2\sin x + \frac{1 - \cos 2x}{2}$$

$$Q = \frac{3}{2}e^{0x} + 2\sin x - \frac{\cos 2x}{2}$$

$$y_p = \frac{1}{D^2+D+1}\left(\frac{3}{2}e^{0x} + 2\sin x - \frac{\cos 2x}{2}\right)$$

$$y_{p_1} = \frac{1}{D^2+D+1}\left(\frac{3}{2}e^{0x}\right)$$

$$= \frac{3}{2}\left[\frac{e^{0x}}{0+0+1}\right]e^{0x} = \frac{3}{2}$$

$$y_{p_2} = \frac{1}{D^2+D+1}2\sin x = 2\left(\frac{1}{D^2+D+1}\sin x\right)$$

$$= 2\left[\frac{1}{-1+D+1}\sin x\right] = \frac{2}{D}\sin x \quad \{D^2=-1\}$$

$$= \quad = -2\cos x$$

$$y_{p_3} = \frac{1}{D^2+D+1}\left(-\frac{\cos 2x}{2}\right) = \frac{-1}{2}\frac{1}{(D^2+D+1)}(\cos 2x)$$

Replace $D^2 = -4$

$$= \frac{-1}{2}\left(\frac{1}{-4+D+1}\cos 2x\right) = \frac{-1}{2}\left[\frac{1}{D-3}(\cos 2x)\right]$$

$$= -\frac{1}{2}\left(\frac{D+3}{D^2-9}\cancel{\sin 2x}\cos 2x\right) \quad (\text{Replace } D^2=-4)$$

$$= \frac{-1}{2}\left(\frac{(D+3)\cos 2x}{-4-9}\right) = \frac{\cancel{-}1}{2}\frac{D\cos 2x + 3\cos 2x}{\cancel{-}13}$$

$$= \frac{2\sin 2x + 6\cos 2x}{26} = \frac{\sin 2x + 6\cos 2x}{26}$$

Now

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

$$= \frac{3}{2} - 2\cos x + \frac{\sin 2x + \overset{3}{\cancel{6}}\cos 2x}{26}$$

$$y = y_c + y_p = e^{-x/2}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{3}{2} - 2\cos x + \frac{2\sin 2x + 3\cos 2x}{26}$$

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Ques $(D^4 - a^4)y = \sin ax$

Sol. The Auxiliary Equation:-

$f(D) = 0$

$D^4 - a^4 = 0$

$(D^2)^2 - (a^2)^2 = 0$

$D + b \qquad\qquad a - b$

$D^2 + a^2 = 0 \qquad\qquad D^2 - a^2 = 0$

$D^2 = -a^2 \qquad\qquad D^2 = a^2$

$D = \pm ia \qquad\qquad D = \pm a$

$y_c = e^{ax}(C_1 + C_2x] + e^{0x}[C_3\cos ax + C_4\sin ax]$

Particular Integrals:

$$y_p = \frac{1}{(D^4 - a^4)} \sin ax$$

$$y_p = \frac{1}{(D^2 - a^2)(D^2 + a^2)} \sin ax$$

$$y_p = \frac{x}{-2a^2(2D)} \sin ax$$

$$= \frac{-1}{4a^2} \int x \cdot \sin ax \, dx$$

$$= \frac{-1}{4a^2}\left[-\frac{x\cos ax}{a} + \int \frac{\cos ax}{a} dx\right]$$

$$= \frac{-1}{4a^2}\left[\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax\right]$$

$$y_p = \frac{1}{4a^3}\left[x\cos ax - \frac{\sin ax}{a}\right]$$

Case-III: $\boxed{Q = x^n}$

Ques 1 $(D^4 + D^3 + D^2)y = 5x^2 + \cos x$

The Auxiliary Equation:

$$F(D) = 0$$

$$D^4 + D^3 + D^2 = 0$$

$$D^2(D^2 + D + 1) = 0$$

$$D^2 = 0 \quad , \quad D^2 + D + 1 = 0$$

$$D = 0, 0 \quad , \quad D = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_c = (C_1 + C_2 x) e^{0x} + e^{-\frac{1}{2}x}\left(C_3 \cos \frac{\sqrt{3}}{2} x + C_4 \sin \frac{\sqrt{3}}{2} x \right)$$

Now,

$$y_{P_1} = \frac{1}{D^4 + D^3 + D^2} \cos x$$

Replace $D^4 = 1^4 = 1$

$D^2 = -(1^2) = -1$

$$y_{P_1} = \frac{1}{1 + D^3 - 1} \cos x = \frac{\cos x}{-a^2 D} = \frac{\cos x}{-1 D}$$

$$= \frac{-\cos x}{D} = -1 \int \cos x \, dx$$

$$y_{P_1} = -\sin x$$

$$y_{P_2} = \frac{1}{D^4 + D^3 + D^2} \cdot 5x^2 = \frac{5x^2}{D^2(1 + D + D^2)}$$

$$= \frac{5}{D^2} \left[1 + (D + D^2)\right]^{-1} x^2$$

$$= \frac{5}{D^2} \left[1 - (D + D^2) + (D + D^2)^2 \cdots \right] x^2$$

$$= \frac{5}{D^2} \left[1 - D - D^2 + D^2 \cdots \right] x^2$$

$$y_{P_2} = \frac{5}{D^2}\left[x^2 - 2x\right]$$

$$= \frac{5}{D}\int (x^2 - 2x)\,dx = \frac{5}{D}\left(\frac{x^3}{3} - x^2\right)$$

$$= 5\int\left(\frac{x^3}{3} - x^2\right)dx$$

$$y_{P_2} = 5\left[\frac{x^4}{12} - \frac{x^3}{3}\right].$$

$$y_P = y_{P_1} + y_{P_2}$$

$$= -\sin x + 5\left(\frac{x^4}{12} - \frac{x^3}{3}\right)$$

Ques ② $(D^4 - 2D^3 + D^2)y = x^3$

The Auxiliary Equation:

$$D^4 - D^3 + D^2 = 0$$

$$D^2(D^2 - D + 1) = 0$$

$D = 0, 0$ $\qquad$ $D^2 - 2D + 1 = 0$

$$(D-1)^2 = 0$$

$$D = 1, 1$$

$$y_c = [C_1 + C_2 x]e^{0x} + [C_3 + C_4 x]e^{x}$$

Now, $y_P = \frac{1}{D^4 - 2D^3 + D^2}\, x^3$

$$= \frac{1}{D^2(\cancel{D^2 - 2D +}(1 - 2D + D^2))}\, x^3$$

$$= \frac{1}{D^2}\left[1 - (2D - D^2)\right]^{-1} x^3$$

$$= \frac{1}{D^2}\left[1 + (2D - D^2) + (2D - D^2)^2 + (2D - D^2)^3 + \dots\right]x^3$$

$$y_p = \frac{1}{D^2}\left[1+2D-D^2+4D^3-4D^3-2D^3 \cdot\right]x^3$$

$$= \cancel{\frac{1}{D^2}\left[x^3+3x^2-6x+6\right]}$$

$$= \cancel{\frac{1}{D}\int (x^3+3x^2-6x+6)\,dx}$$

$$= \frac{1}{D^2}\left[1+2D+3D^2+4D^3\right]x^3$$

$$= \frac{1}{D^2}\left[x^3+6x^2+18x+24\right]$$

$$y_p = \frac{1}{D}\int (x^3+6x^2+18x+24)\,dx$$

$$= \frac{1}{D}\left(\frac{x^4}{4}+\frac{6x^3}{3}+\frac{18x^2}{2}+24x\right)dx$$

$$= \int\left(\frac{x^3}{3}+2x^3+9x^2+24x\right)dx$$

$$y_p = \frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

$$y = y_c + y_p$$

$$= \left(\left[c_1+c_2x\right]e^{0x}+\left[c_3+c_4x\right]e^{x}\right)+\left[\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2\right]$$

Que (3) $(D-2)^2 y = e^{2x}+\sin 2x + x^2$

The Auxiliary equation

$$(D-2)^2 = 0$$

Consider $D-2=0$

$$D = 2, 2$$

$$y_c = \left[c_1+c_2x\right]e^{2x}$$

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$$y_{P_1} = \frac{1}{(D-2)^2} e^{2x} = \frac{x \cdot e^{2x}}{2(D-2)}$$

$$= \frac{x^2 \cdot e^{2x}}{2}$$

$$y_{P_2} = \frac{1}{(D-2)^2} \sin 2x$$

$$= \frac{1}{D^2-4D+4} \sin 2x$$

$$D^2 = -4$$

$$= \frac{1}{-4-4D+4} \sin 2x$$

$$= \frac{1}{-4D} \sin 2x$$

$$= \frac{-1}{4} \int \sin 2x \, dx$$

$$= \frac{-1}{4} \int 2 \sin x \cos x \, dx$$

$$= \frac{-1}{4}\left[\frac{-\cos 2x}{2}\right]$$

$$y_{P_2} = \frac{\cos 2x}{8}$$

$$y_{P_3} = \frac{1}{D^2-4D+4} \cdot x^2 = \frac{1}{4\left(1-D+\frac{D^2}{4}\right)} x^2$$

$$= \frac{1}{4}\left[1-\left(D-\frac{D^2}{4}\right)\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1+\left(D-\frac{D^2}{4}\right)+\left(D-\frac{D^2}{4}\right)^2\right] \cdot x^2$$

$$= \frac{1}{4}\left[1+D-\frac{D^2}{4}+D^2 \ldots\right] x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3D^2}{4} \ldots\right] x^2$$

$$y_{P_3} = \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}\right]$$

$$y_P = y_{P_1} + y_{P_2} + y_{P_3}$$

$$= \frac{x^2 \cdot e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{4}\right)$$

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Ques 5 $(D^4 + 8D^2 + 16)y = \sin^2 x$

The Auxiliary equation:

$$f(D)y = 0$$

$$D^4 + 8D^2 + 16 = 0$$

$$(D^2+4)^2 = 0$$

Consider, $D^2 + 4 = 0$

$$D = \sqrt{-4}$$

$$D = \pm 2i, \pm 2i$$

$$y_c = e^{0x}\left[(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x\right]$$

Now, $y_P = \frac{1}{(D^4 + 8D^2 + 16)} \sin^2 x$

$$= \frac{1}{(D^4+8D^2+16)}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^4+8D^2+16)} \frac{-\cos 2x}{2}\right]$$

$$= \frac{-1}{4}\left[\frac{1}{(D^4+8D^2+16)}(\cos 2x)\right]$$

Replace $D^4 = 2^4$

$$y_P = \frac{-1}{4}\left[\frac{1}{32+8D^2}(\cos 2x)\right]$$

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$$y_p = \frac{1}{-4}\left[\frac{x}{16D}\cos 2x\right]$$

$$= \frac{-1}{64}\int x\cos 2x \pm \frac{1}{64}(2+x^2\cos 2x)$$

$$y_p = \cancel{\frac{-1}{64}\frac{\sin 2x}{2}} = \frac{-1}{128}(\sin 2x)\ \frac{1}{64}(2+x^2\cos 2x)$$

Ques 9
Sol.

$(D^2+4)y = \cos 2x$

The Auxiliary Equation

$$F(D) = 0$$

$$D^2+4 = 0$$

$$D^2 = -4$$

$$D = \sqrt{-4} = \pm 2i$$

$$\cancel{D^2+4+4D-4D}$$

$$\cancel{(D^2+2)^2 - 2D}$$

$$y_c = C_1\cos 2x + C_2\sin 2x$$

$$y_p = \frac{1}{D^2+4}\cos 2x$$

∴ At $D^2 = -4$, Deno becomes zero.

$$\therefore y_p = \frac{x}{2D}\cos 2x$$

$$= \frac{x}{2}\int \cos 2x$$

$$y_p = \frac{x}{4}\sin 2x$$

$$y = y_c + y_p$$

$$= C_1\cos 2x + C_2\sin 2x + \frac{x}{4}\sin 2x$$

Case V

Que. 1 $(D^2-1)y = x\cdot \sin 3x$

The A.E $\Rightarrow$ $D^2-1=0$

$D = \pm 1$

$y_c = C_1e^{x} + C_2e^{-x}$

Now, $y_p = \frac{1}{(D^2-1)} x\cdot\sin 3x$

$= \frac{1}{(D^2-1)}[x\cdot\sin 3x]$

$= \left[x - \frac{1}{(D^2-1)}\cdot 2D\right]\frac{1}{(D^2-1)} \sin 3x \quad \{D^2=-9\}$

$= \left[x - \frac{1}{(D^2-1)}\cdot 2D\right]\frac{1}{-10}\sin 3x$

$= \frac{-1}{10}\left[x\sin 3x - \frac{1}{(D^2-1)} 2(D\sin 3x)\right]$

$= \frac{-1}{10}\left[x\sin 3x - \frac{1}{D^2-1} 2\ \cancel{3\sin}(3\cos 3x)\right]$

$= \frac{-1}{10}\left[x\sin 3x - 6\frac{1}{(D^2-1)}\cos 3x\right] \{D^2=-9\}$

$y_p = \frac{-1}{10}\left[x\sin 3x + 6\frac{1}{(10)}\cos 3x\right]$

$\therefore y = y_c + y_p$

$y = C_1e^{x} + C_2e^{-x} + \left[\frac{-1}{10}\left(x\sin 3x + \frac{6}{10}\cos 3x\right)\right]$

Ques 2. $(D^2-2D+1)y = xe^x \sin x$

The Auxiliary equation:

$$D^2-2D+1=0$$

$$D=1$$

$$\therefore y_c = (C_1+C_2x)e^x$$

$$y_p = \frac{1}{(D-1)^2} x \cdot e^x \cdot \sin x$$ $\left\{\because e^x \text{ is given we can use case} \cdot 4\right\}$

$$= e^x \frac{1}{[D+1-1]^2} x \cdot \sin x$$

$$= e^x \frac{1}{D^2} x \cdot \sin x$$

$$= e^x \frac{1}{D} \int x \cdot \sin x = e^x \frac{1}{D}\left[x(-\cos x)+\sin x\right]$$

$$= e^x\left[\int(-x\cos x) + \int \sin x\right]$$

$$= e^x\left[-(+x\sin x + \cos x) + (-\cos x)\right]$$

$$= e^x\left[-x\sin x - \cos x - \cos x\right]$$

$$y_p = -e^x\left[x\sin x + 2\cos x\right]$$

$$y = y_c + y_p$$

$$y = (C_1+C_2x)e^x + \left[-e^x(x\sin x + 2\cos x)\right]$$

(Don't use Case V)

Ques: 3 $(D^2+1)y = x^2 \sin 2x$

The Auxiliary Equation:

$$D^2+1=0$$
$$D^2=-1$$
$$D=\pm i$$

$$y_c = C_1\cos x + C_2\sin x$$

$$y_p = \frac{1}{D^2+1} x^2 \sin 2x.$$

$$y_p = \frac{1}{D^2+1} \text{ Imag part of } (x^2 \cdot e^{i2x})$$

$$\left\{ \begin{array}{l} e^{i2x} = \cos 2x + i\sin 2x \\ \therefore \sin 2x = \text{I.P. of } e^{i2x} \end{array} \right\}$$

$$y_p = e^{i2x}\left[\frac{1}{(D+2i)^2+1} \cdot x^2\right]$$

$$= e^{i2x}\left[\frac{1}{D^2+4Di-3} \cdot x^2\right]$$

$$= \frac{-1}{3} e^{i2x}\left[\frac{1}{\left(1-\frac{4Di}{3}-\frac{D^2}{3}\right)} \cdot x^2\right]$$

$$y_p = \frac{-1}{3} e^{i2x}\left\{\left[1-\left(\frac{4Di}{3}+\frac{D^2}{3}\right)\right]^{-1} \cdot x^2\right\}$$

$$= \frac{-e^{i2x}}{3}\left\{\left[1+\left(\frac{4Di}{3}+\frac{D^2}{3}\right)+\left(\frac{4Di}{3}+\cdots\right)^2\right] x^2\right\}$$

$$y_p = \frac{-e^{i2x}}{3}\left[1+\frac{4Di}{3}+\frac{D^2}{3}-\frac{16D^2}{9}\right] \cdot x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{4Di}{3}-\frac{13D^2}{9}\right] x^2$$

$$y_{P_1} = \frac{-e^{i2x}}{3}\left[x^2 + \frac{4i}{3}(2x) - \frac{13\times 2}{9}\right]$$

$$y_{P_1} = \frac{-e^{2ix}}{3}\left[x^2 + \frac{8ix}{3} - \frac{26}{9}\right]$$

Now,

y_P = Imaginery part of y_{P_1}

$$= \text{I.P. of} \left(\frac{-e^{2ix}}{3}\right)\left[x^2 - \frac{26}{9} + \frac{8xi}{3}\right]$$

$$= \frac{-1}{3}\ \text{I.P. of} (\cos 2x + i\sin 2x)\left[\left(x^2 - \frac{26}{9}\right) + \frac{8xi}{3}\right]$$

Que 4 $(D^4-1)y = \cos x \cdot \cosh x$

The Auxiliary Equation:

$$D^4-1 = 0$$
$$D^4 = 1$$
$$D = \pm 1, \pm i$$

$$\therefore y_c = C_1e^{x} + C_2e^{-x} + C_3\cos x + C_4\sin x$$

$$y_p = \frac{\cos\left(\frac{e^{x}}{2} + \frac{e^{-x}}{2}\right)}{D^4-1}$$

$$y_{p_1} = \frac{e^{x}}{2} \cdot \frac{\cos x}{D^4-1} = \frac{e^{x}}{2} \cdot \frac{\cos x}{(D^4-1)}$$

$$= \frac{e^{x}}{2} \cdot \frac{\cos x}{(D+1)^4-1} = \frac{e^{x}}{2} \cdot \frac{\cos x}{(D^4-4D^3+6D^2-4D+1)-1}$$

$$y_{p_1} = \frac{e^{x}}{2} \cdot \frac{\cos x}{\cancel{\cos}(-5)} = \frac{-e^{x}}{10}\cos x$$

$$y_{p_2} = \frac{e^{-x}}{2} \cdot \frac{\cos x}{(D^4-1)}$$

$$= \frac{e^{-x}}{2} \cdot \frac{\cos x}{(D-1)^4-1}$$

$$= \frac{e^{-x}}{2} \cdot \frac{\cos x}{(1+4D-6-4D)}$$

$$y_{p_2} = \frac{e^{-x}}{2} \cdot \frac{\cos x}{(-5)} = \frac{-e^{-x}\cos x}{10}$$

$$y_p = \frac{-\cos x}{5}\left(\frac{e^{x} + e^{-x}}{2}\right)$$

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Ques. 10 (ii) $(D^2-3D+2)y = 2e^x \sin\frac{x}{2}$

The Auxiliary Equation:

$$D^2-3D+2=0$$

$$D=+2,1$$

$$y_c = C_1e^x + C_2e^{+2x}$$

$$y_p = \frac{2\ e^x \sin(x/2)}{D^2-3D+2}$$

$$= 2\cdot e^x \cdot \frac{1}{(D+1)^2-3(D+1)+2} \sin(x/2)$$

$$= 2\cdot e^x \cdot \frac{\sin(x/2)}{(D^2+2D+1-3D-3+2)}$$

$$= 2e^x \frac{\sin(x/2)}{(D^2-D)}$$

$$= 2\cdot e^x \frac{1}{-1/4-D} \sin(x/2)$$

$$= -8e^x \frac{1}{4D+1} \sin(x/2)$$

$$= -8\cdot e^x \frac{(4D-1)\sin(x/2)}{16D^2-1}$$

$$= -8e^x\left[\cos(x/2) - \sin x/2\right]$$

$$= -2e^x \frac{1}{D+1/4} \sin(x/2)$$

$$= -2\cdot e^x \frac{(D-1/4)(\sin x/2)}{(D^2-1/16)}$$

$$= -2\cdot e^x \frac{(\cos x/2)\cdot 1/2 - 1/4 \sin(x/2)}{\left(-\frac{1}{4}-\frac{1}{16}\right)}$$

$$= \frac{32}{5}e^{x}\left[\frac{1}{2}\cos\frac{x}{2} - \frac{1}{4}\sin\frac{x}{2}\right]$$

$$= \frac{32e^{x}}{5}\cdot\frac{1}{4}\left[2\cos(x/2) - \sin(x/2)\right]$$

$$y_p = \frac{8}{5}e^{x}\left[2\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]$$

Ques 13 $(D^2 - 7D - 6)y = (1+x^2)e^{2x}$

The Auxiliary Equation:

$$D = \frac{7+\sqrt{73}}{2}, \frac{7-\sqrt{73}}{2}$$

$$y_c = C_1 e^{\left(\frac{7+\sqrt{73}}{2}\right)x} + C_2 e^{\left(\frac{7-\sqrt{73}}{2}\right)x}$$

Now $y_p = \dfrac{e^{2x}(1+x^2)}{(D^2-7D-6)}$

$$= e^{2x}\frac{(1+x^2)}{(D+2)^2 - 7(D+2) - 6}$$

$$= \frac{e^{2x}(1+x^2)}{D^2 - 3D - 16}$$

$$= \frac{e^{2x}}{-16}\left[\frac{(1+x^2)}{1 + 3D/16 - D^2/16}\right]$$

$$= \frac{-e^{2x}}{16}\left[1 + \left(\frac{3D}{16} - \frac{D^2}{16}\right)\right]^{-1}(1+x^2)$$

$$= \frac{-e^{2x}}{16}\left[1 - \frac{3D}{16} + \frac{D^2}{16} + \frac{9D^2}{256}\right](1+x^2)$$

$$= \frac{-e^{2x}}{16}\left[(1+x^2) - \frac{3}{16}2x - \frac{25\times 2}{256}\right]$$

$$y_p = \frac{e^{2x}}{-16}\left[1+x^2 - \frac{3x}{16} - \frac{50}{256}\right] \quad 79$$

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Ques. (2) $(D^2+2D+1)y = x \cdot e^{-x}\cos x$

The Auxiliary Equation:

$D^2+2D+1=0$

$D=-1.$

$\therefore y_c = e^{-x}(C_1+C_2x)$

Now, $y_p = \dfrac{x \cdot e^{-x}\cos x}{D^2+2D+1}$

$= e^{-x}\dfrac{1}{(D-1)^2+2(D-1)+1} x \cdot \cos x$

$= e^{-x}\dfrac{x\cos x}{D^2}$

$= e^{-x}\dfrac{1}{D}\int x \cdot \cos x$

$= e^{-x}\dfrac{1}{D}[x\sin x+\cos x]$

$= e^{-x}\int [x\sin x+\cos x]$

$= e^{-x}[x(-\cos x)+\sin x+\sin x]$

$y_p = e^{-x}[-x\cos x+2\sin x]$

$y_p = -e^{-x}[x\cos x-2\sin x]$

Ques (2) $(D^2-1)y = x\sin x + e^x + x^2e^x$

The Auxiliary equation:

$D^2-1=0$

$D=\pm 1$

$y_c = C_1e^x + C_2e^{-x}$

Now. $y_{P_1} = \dfrac{x \sin x}{D^2-1}$

$$= \left[x - \frac{1}{(D^2-1)} 2D\right] \frac{1}{(D^2-1)} \sin x$$

$$= \left[x - \frac{1}{(D^2-1)}(2D)\right] \frac{-1}{2} \sin x$$

$$= \left[\frac{-x}{2}\sin x + \frac{1}{2(D^2-1)} 2\cos x\right]$$

$$y_{P_1} = \left[\frac{-x}{2}\sin x + \frac{\cos x}{(-2)}\right]$$

$$y_{P_2} = \frac{e^x}{D^2-1} = \frac{x e^x}{2D}$$

$$y_{P_2} = \frac{x}{2} e^x$$

$$y_{P_3} = \frac{x^2 \cdot e^x}{D^2-1} = e^x \frac{x^2}{(D+1)^2-1}$$

$$= e^x \cdot \frac{x^2}{D^2+2D}$$

$$= \frac{e^x}{2D} \frac{x^2}{\left[1+\frac{D^2}{2}\right]} = \frac{e^x}{2D}\left[1+\frac{D^2}{2}\right]^{-1} x^2$$

$$= \frac{e^x}{2D}\left[1-\frac{D^2}{2}\right] x^2$$

$$= \frac{e^x}{2D} \int (x^2-1)$$

$$y_{P_3} = \frac{e^x x^3}{6} - \frac{e^x \cdot x}{3}$$

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Ques 2. $(D^2-4D+4)y = \frac{e^{2x}}{1+x^2}$

Sol. The Auxiliary Equation:

$$(D-2)^2 = 0$$

$$D = 2, 2$$

$$y_c = (C_1+C_2x)e^{2x}.$$

$$y_p = \frac{1}{(D-2)^2} \frac{e^{2x}}{1+x^2}$$

$$= e^{2x} \frac{1}{(D+2-2)^2} \frac{1}{1+x^2} = e^{2x} \frac{1}{D^2} \frac{1}{(1+x^2)}$$

$$= e^{2x} \frac{1}{D} \int \frac{1}{1+x^2} = e^{2x} \frac{1}{D} (\tan^{-1}x)$$

$$= e^{2x} \int \underset{u}{\tan^{-1}x} \cdot \underset{v}{1}\, dx$$

$$= e^{2x} \left\{ \tan^{-1}x \int 1 dx - \int \left[\int 1 dx \frac{d}{dx} \tan^{-1}x \right] dx \right\}$$

$$= e^{2x} \left\{ \left[\tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2} dx \right\} \right.$$

$$= e^{2x} \left[x\tan^{-1}x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \right]$$

$$y_p = e^{2x} \left[x\tan^{-1}x - \frac{1}{2} \log(1+x^2) \right]$$

Ques 3. $(D^2+3D+2)y = \sin(e^x)$

The A.E: $(D+2)(D+1) = 0$

$$D = -2, -1.$$

$$y_p = \frac{1}{(D+2)(D+1)} \sin(e^x)$$

$$= \frac{1}{(D+2)}\left[\frac{1}{(D+1)} \sin(e^x)\right]$$

$$= \frac{1}{D+2}\left[e^{-x}\int e^x \sin(e^x)\,dx\right.$$

Put $e^x = t$

$e^x dx = dt$

$$y_p = \frac{1}{(D+2)}\left[e^{-x}\int \sin t\,dt\right]$$

$$= \frac{1}{D+2}\left[e^{-x}(-\cos t)\right]$$

$$= \frac{-1}{D+2}\left[e^{-x}\cos e^x\right]$$

$$y_p = -e^{-2x}\int e^{2x}\, e^{-x}\cos(e^x)\,dx$$

$$= -e^{-2x}\int e^x \cos(e^x)\,dx$$

Put $e^x = t$

$e^x dx = dt$

$$y_p = -e^{-2x}\int \cos t\,dt$$

$$y_p = -e^{-2x}\sin(e^x)$$

Ques 4 $(D^2+5D+6)y = e^{-2x}\sec^2 x\,(1+2\tan x)$

#) $\int e^x \cdot [f(x)+f'(x)]\,dx = e^x \cdot f(x)$ } #

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$$y_p = \frac{1}{(D+3)(D+2)} e^{-2x} \sec^2 x\,(1+2\tan x)$$

$$= e^{-2x} \frac{1}{(D-2+3)(D+2-2)} \sec^2 x\,(1+2\tan x)$$

$$= e^{-2x} \frac{1}{(D+1)} \frac{1}{D} [\sec^2 x\,(1+2\tan x)] \Rightarrow \sec^2 x + 2\tan x \sec$$

$$= e^{-2x} \frac{1}{D+1} \left[\tan x + 2\frac{(\tan x)^2}{2}\right]$$

$$= e^{-2x} \cdot e^{-x} \int e^x [\tan x - 1 + \sec^2 x]\,dx$$

$$= e^{-3x} \cdot e^x (\tan x - 1)$$

$$y_p = e^{-2x}(\tan x - 1)$$

Cauchy's L.D.E with Variable Coefficients:

Ques 1 $$x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x)$$

which is Cauchy's L.D.E

Put $x = e^z$

$\log x = z$

$$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y \quad \left[D = \frac{d}{dx}\right]$$

The D.E equation:

$D(D-1)y - D\ldots$

Cauchy's L.D.E with Variable Coefficients:

Ques 1 $$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x)$$

which is Cauchy's L.D.E

Put $x = e^z$

$\log x = z$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \left[D = \frac{d}{dx}\right]$$

The D.E equation:

$$D(D-1)y - Dy + 4y = (\cos z)$$

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$(D^2 - D - D + 4)y = \cos z$

$(D^2 - 2D + 4) = \cos z$

The Auxiliary equation:

$D^2 - 2D + 4 = 0$

$D = 1 \pm \sqrt{3}i$

$\therefore \; y_c = e^z(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z)$

Now $y_p = \dfrac{1}{(D^2 - 2D + 4)} \cos z \qquad (D^2 = -1)$

$= \dfrac{1}{(-1 - 2D + 4)} \cos z = \dfrac{1}{(-2D + 3)} \cos z$

$= \dfrac{(2D + 3)}{(-2D + 3)(+2D + 3)} \cos z$

$= \dfrac{3\cos z - 2\sin z}{9 - 4D^2} \qquad (D^2 = -1)$

$y_p = \dfrac{3\cos z - 2\sin z}{13}$

$y = y_c + y_p$

$= x(c_1 \cos\sqrt{3}\log x + c_2 \sin\sqrt{3}\log x) + \left[\dfrac{3\cos(\log x) - 2\sin(\log x)}{13}\right]$

$y = x\left[c_1 \cos\sqrt{3}(\log x) + c_2 \sin\sqrt{3}(\log x)\right] + \dfrac{1}{13}\left[3\cos(\log x) - 2\sin \log x\right]$

Ques 2. $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 \log x$

(powers don't match)

Multiply by x

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3 \log x$$

Put $x = e^z$

$$\log x = z \qquad \left[D = \frac{d}{dx} \right]$$

$$x\frac{dy}{dx} = Dy$$

$$D(D-1)(D-2) + 3[D(D-1)] + Dy = e^{3z}.z$$

$$[D^3 \cancel{-3D^2} + \cancel{2D} + \cancel{3D^2} \cancel{-3D} + \cancel{D}]y = e^{3z}.z$$

$$D^3y = e^{3z}.z$$

The Auxiliary equation

$$D^3 = 0$$

$$D = 0,0,0$$

$$y_c = (C_1 + C_2z + C_3z^2)e^{0z}$$

Now $y_p = \dfrac{1}{D^3}(e^{3z}.z)$

$$= \frac{1}{D^2}\int e^{3z}.z\,dz$$

$$= \frac{1}{D^2}\left[\frac{ze^{3z}}{3} - \frac{e^{3z}}{9}\right]$$

$$= \frac{1}{D}\left\{\frac{1}{3}\left[\frac{ze^{3z}}{3} - \frac{e^{3z}}{9}\right] - \frac{e^{3z}}{27}\right\}$$

$$= \frac{1}{9}\left[\frac{ze^{3z}}{3} - \frac{e^{3z}}{9}\right] - \frac{e^{3z}}{27(3)} - \frac{1}{27}\frac{e^{3z}}{3}$$

$$= \frac{1}{27}\left(e^{3z}.z - e^{3z}\right)$$

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$$y_p = \frac{1}{27} e^{3z}(z-1)$$

$$y_p = \frac{1}{27} x^3 (\log x - 1)$$

$$\therefore y = y_c + y_p$$
$$= [C_1 + C_2 \log x + C_3 (\log x)^2] + \frac{x^3}{27}(\log x - 1)$$

May-17 Ques

$$x^2 \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4 \log x$$

Multiply by x on both sides

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4x \log x$$

Put $x = e^z$

$\log x = z$

$$x\frac{dy}{dx} = Dy$$

$$D(D-1)(D-2) + 3[D(D-1)] + Dy + y = 4e^z z$$

$$[D^3 - 3D^2 + 2D + 3D^2 - 3D + D + 1]y = 4e^z \cdot z$$

$$[D^3 - \cancel{3D^2} + \cancel{3D^2} - \cancel{3D} + \cancel{3D} + 1]y = 4ze^z$$

$$[D^3 + 1]y = 4ze^z$$

The Auxiliary equation:-

$D^3 + 1 = 0$

$D^3 = -1$

$$D = -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$y_c = e^{-z}\left[C_1 \sin\left(1 + \frac{\sqrt{3}}{2}\right) + C_2 \cos\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right]$$

$$y_p = \frac{1}{(D^3+1)}\left[4e^z \cdot z\right]$$

$$= e^z \frac{1}{(D+1)^3+1} 4 \cdot z$$

$$= e^z \frac{1}{D^3+1+1+3D^2+3D} 4 \cdot z$$

$$= 4e^z \left[\frac{1}{-D-3+2+3D} z\right]$$

$$= 4e^z \frac{1}{2D-1} z$$

$$= 4e^z \frac{(D-2)z}{(D+2)(D-2)}$$

$$= 4e^z \frac{D - 2z}{D^2-4}$$

$$= 4e^z \frac{Dz - 2z}{1-4}$$

$$= \frac{-4}{3} e^z \left[1 - 2z\right]$$

$$= \frac{-4}{3} x \left[1 - 2\log x\right]$$

$$y_p = 4e^z \frac{(2D+1)z}{4D^2-1}$$

$$= 4e^z \frac{2Dz + z}{-5}$$

$$= \frac{-4e^z}{5}(2+z)$$

$$y_p = \frac{-4}{5} x\left[2 + \log x\right]$$

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Ques 3 $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

Sol Put $x = e^z$

$\log x = z$

$x\frac{dy}{dx} = Dy$

$D(D-1) - 3Dy + 5y = e^{2z}\sin z$

$D^2 - D - 3Dy + 5y = e^{2z}\sin z$

$[D^2 - D - 3D + 5]y = e^{2z}\sin z$

The Auxiliary equation:

$D^2 - 4D + 5]y$

$D = 2+i, 2-i$

$\therefore y_c = e^{2z}[C_1\sin z + C_2\cos z]$

Now $y_p = \frac{1}{D^2-4D+5} e^{2z}\sin z$

$= e^{2z}\left[\frac{1}{(D+2)^2 - 4(D+2) + 5}\right]\sin z$

$e^{2z}\left[\frac{1}{D^2 + 4D + 4 - 4D - 8 + 5}\right]\sin z$

$= e^{2z}\left[\frac{1}{D^2+1}\right]\sin z$

$= e^{2z}\left[\frac{z}{2D}\sin z\right]$

$= e^{2z}\frac{z}{2}\int\sin z$

$y_p = -e^{2z}\frac{z}{2}\cos z$

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$y = y_c + y_p$

$= x^2[c_1 \sin(\log x) + c_2 \cos(\log x)] + \{-\frac{x^2}{2}[\log x \cos \log$

$= x^2[c_1 \sin(\log x) + c_2 \cos(\log x)] - \frac{x^2}{2}\{\log x \cos(\log x)$

Ques ① $(D^2 + D)y = \frac{1}{1+e^x}$

The Auxiliary equation:

$D(D+1) = 0$

$D = 0, -1$

$y_c = c_1 + c_2 e^{-x}$

$y_p = \frac{1}{D(D+1)} \frac{1}{1+e^x}$

$= \frac{1}{D}\left[\frac{1}{(D+1)} \frac{1}{(1+e^x)}\right]$

$= \frac{1}{D}\left[e^{-x} \int e^x \frac{1}{1+e^x} dx\right]$

$\log(1+e^x) \int e^{-x} dx - \int e^x dx \frac{d}{dx} \log(1+e^x)]$

$= \log(1+e^x) - e^{-x} - \int \frac{-e^x}{1+e^x} dx$

$y_p = \log(1+e^x)(e^x) - \log(1+e^x)$

$y_p = -e^{-x} \log(1+e^x) - \log(1+e^x)$

Legendre's L.D.E

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Que 1 $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

Put $(3x+2) = e^z \Rightarrow \therefore \log(3x+2) = z$

$x = \frac{e^z - 2}{3}$

$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1) = (9D^2 - 9D)y$

$3(3x+2)\frac{dy}{dx} = 3(3)Dy = 9Dy$

The Given D.E becomes,

$(9D^2-9D)y + 9Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$

$9(D^2 - D + D - 4)y = \frac{e^{2z} - \cancel{4e^z} + 4 + \cancel{4e^z} - 8 + 3}{3}$

$(D^2-4)y = \frac{1}{27}(e^{2z} - 1)$

$D^2 - 4 = 0$

$D = \pm 2$

$\# \left[y_c = c_1 e^{2z} + c_2 e^{-2z} \right] \#$

Now, $y_p = \frac{1}{27}\left[\frac{1}{D^2-4} e^{2z} - \frac{1}{D^2-4} e^{0z}\right]$

$\frac{1}{27}\left[\frac{z}{2D} e^{2z} - \frac{1}{-4}\right]$

$= \frac{1}{27}\left[\frac{z}{2}\int(e^{2z}) + \frac{1}{4}\right]$

$= \frac{1}{27}\left[\frac{z}{2}\frac{e^{2z}}{2} + \frac{1}{4}\right]$

$y_p = \frac{1}{27}\left[\frac{ze^{2z}}{4} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z}+1)$

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$$y = y_c + y_p$$

$$y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}\left[\frac{\log(3x+2)}{(3x+2)^2+1}\right]$$

#D-16

Ques 2

$$(1+2x)^2\frac{d^2y}{dx^2} - 3(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$$

Sol.

Put $(1+2x) = e^z$

$$(1+2x)^2\frac{d^2y}{dx^2} = 2^2 D(D-1) = (4D^2-4D)y$$

$$6(1+2x)\frac{dy}{dx} = 3.2Dy = 6Dy$$

The D.E becomes,

$$[(4D^2-4D)y - 6Dy + 16y] = 8e^{2z}$$

$$(4D^2 - 4D - 3.2D + 16)y = 8e^{2z}$$

$$2(2D^2 - 5D - 3D + 8)y = 8e^{2z}$$

$$(2D^2 - 5D + 8)y = 4e^{2z}$$

The Auxiliary equation

$$2D^2 - 5D + 8 = 0$$

$$D = \frac{5}{2} \pm \frac{\sqrt{39}i}{4}$$

$$y_c = e^{4z}\left[C_1\cos\left(\frac{\sqrt{39}}{4}\right) + C_2\sin\left(\frac{\sqrt{39}}{4}\right)\right]$$

Now $y_p = \frac{1}{2D^2-5D+8}4e^{2z}$

$$= 4\left[\frac{1}{2D^2-5D+8}e^{2z}\right]$$

$$= 4\left[\frac{1}{8-5D+8}e^{2z}\right]$$

$$y_p = \frac{-4}{5}\left[\frac{1}{D}e^{2z}\right]$$

$$y_p = \frac{-4}{5}\frac{e^{2z}}{2} = \frac{-4e^{2z}}{10}$$

Now, $y = y_c + y_p$

$$= (1+2x)^4\left[c_1\cos\frac{\sqrt{39}}{4} + c_2\sin\frac{\sqrt{39}}{4}\right]$$

$$- \frac{4}{10}(1+2x)^2$$

$$= (1+2x)^2\left\{c_1\cos\frac{\sqrt{39}}{4} + c_2\sin\frac{\sqrt{39}}{4}\right\}$$

$$-\frac{4}{10}(1+2x)^2$$

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Variation of Parameters

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Ques 1 $(D^2+1)y = \sec x \tan x$

The Auxiliary equation:

$$D^2+1=0$$

$$D^2=-1$$

$$D=\pm i$$

$$y_c = C_1\cos x + C_2 \sin x$$

Let $y_p = uy_1 + vy_2$

$$y_1 = \cos x \;,\; y_2 = \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\sin^2 x + \cos^2 x)$$

$$W = 1$$

$$\therefore \; u = -\int \frac{y_2 Q}{W}\,dx = -\int \sin x . \sec x \tan x \, dx$$

$$= -\int \tan^2 x\, dx = -\int (\sec^2 x - 1)\,dx$$

$$= -(\tan x - x)$$

$$v = \int \frac{y_1 Q\,dx}{W} = \int \frac{\cos x}{1} \sec x \tan x \, dx$$

$$= \int \tan x \, dx$$

$$v = \log(\sec x)$$

$$y_p = uy_1 + vy_2 = (x - \tan x)\cos x + \log(\sec x)\sin x$$

$$y = y_c + y_p$$

$$= C_1\cos x + C_2 \sin x + (x - \tan x)\cos x + \log(\sec x)\sin x$$

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Ques 8 $(D^2-1)y = \frac{2}{1+e^x}$

The Auxiliary equation:

$D^2-1=0$

$D^2=1$

$D=\pm 1$

$y_c = C_1e^x + C_2e^{-x}$

$y_p = uy_1 + vy_2$

Here $y_1 = e^x$, $y_2 = e^{-x}$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -(e^{-x}\times e^x) - e^{-x}\times e^x$
$= -1-1 = -2$

$\therefore u = -\int \frac{y_2 Q}{W}dx = -\int \frac{e^{-x}}{-2}\left[\frac{2}{1+e^x}\right]dx$

$= +\int \frac{e^{-x}}{\cancel{2}} \frac{\cancel{2}}{1+e^x}dx$ [Multiply and divide by e^{-x}]

* [$u = \cancel{\log(1+e^{x})} \int \frac{e^{-2x}dx}{1+e^{-x}}$] *

$v = \int \frac{y_1 Q}{W}dx = \int \frac{e^x}{-2}\frac{2}{1+e^x}dx$

$= \frac{-\cancel{2}}{\cancel{2}}\int \frac{e^x}{1+e^x}dx$

$= \frac{-\cancel{2}}{\cancel{2}}\log(1+e^{+x}) = \frac{-\cancel{2}}{\cancel{2}}\log(1+e^x)$

$= -\log(1+e^x)$

Remaining steps of $u(x)$

Put $e^{-x} = t$

$$-e^{-x}\,dx = dt$$

$$\therefore\ u = -\int \frac{t\,dt}{t+1} = -\int \left[\frac{t+1-1}{t+1}\right] dt$$

$$= -\int \left(1 - \frac{1}{t+1}\right) dt$$

$$= -\left[t - \log(t+1)\right]$$

$$= \log(t+1) - t$$

$$\left[u = \log(e^{-x}+1) - e^{-x}\right]$$

$$y_p = u y_1 + v y_2$$

$$= \left[\log(e^{-x}+1) - e^{-x}\right] e^{x} + \left[\log(1+e^{x})(e^{-x})\right]$$

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Ques 3. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

Given eqn is Cauchy's D.E

Put $x = e^z \Rightarrow z = \log x$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$$

$$x\frac{dy}{dx} = Dy \qquad \left[\text{where } D = \frac{d}{dz}\right]$$

lly, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

The D.E becomes,

$$(D(D-1) - 3D + 5)y = e^{2z}\sin z$$

$$(D^2 - 4D + 5)y = e^{2z}\sin z$$

The Auxiliary equation:

$$D^2 - 4D + 5 = 0$$

$$D = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$y_c = e^{2z}(c_1\cos z + c_2\sin z)$$

$$y_p = \frac{1}{(D^2 - 4D + 5)} e^{2z}\sin z = e^{2z}\frac{1}{(D+2)^2 - 4(D+2) + 5}\sin z$$

$$= e^{2z}\frac{1}{D^2 + \cancel{4D} + 4 - \cancel{4D} + 8 + 5}\sin z = e^{2z}\frac{1}{D^2+1}\sin z$$

$$y_p = ze^{2z}\frac{1}{2D}\sin z = \frac{-ze^{2z}}{2}\cos z$$

$$y = y_c + y_p$$

$$= x^2[c_1\cos(\log x) + c_2\sin(\log x)] - \frac{x^2}{2}\log x\cos(\log x)$$

Ques 8 $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\frac{\log x}{x^2}$

Sol. Let $x=e^z \Rightarrow z=\log x$

$$\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx}=Dy \quad \left[\text{where } \frac{d}{dz}=D\right]$$

lly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Given DE becomes,

$$(D^2-D+5D+3)y=e^{-2z}z$$

$$(D^2+4D+3)y=e^{-2z}\cdot z$$

The Auxiliary equation:

$$D^2+4D+3=0$$

$$(D+3)(D+1)=0$$

$$D=-1,-3$$

$$y_c=C_1e^{-z}+C_2e^{-3z}$$

$$y_c=C_1\frac{1}{x}+C_2\frac{1}{x^3}$$

$$y_p=\frac{1}{D^2+4D+3}e^{-2z}\cdot z=e^{-2z}\frac{1}{(D-2)^2+4(D-2)+3}z$$

$$=e^{-2z}\frac{1}{D^2-4D+4D-8+3+4}z$$

$$=e^{-2z}\frac{1}{D^2-1}z=-e^{-2z}(1-D^2)^{-1}z$$

$$=-e^{-2z}[1+D^2+D^4+\cdots]z$$

$$y_p=-e^{-2z}z=-\frac{\log x}{x^2}$$

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$$y = y_c + y_p$$
$$= C_1 \frac{1}{x} + C_2 \frac{1}{x^3} - \frac{\log x}{x^2}$$

Ques 3. $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x$

Sol.

Put $5+2x = e^{z}$

$z = \log(5+2x)$

and $x = \frac{e^{z}-5}{2}$

$(5+2x)\frac{dy}{dx} = 2Dy$ $\left[\text{where, } \frac{d}{dz} = D\right]$

llly, $(5+2x)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$

Given D.E becomes,

$$4(D^2-D)y - 6\times 2Dy + 8y = 6\left(\frac{e^{z}-5}{2}\right)$$

$$(D^2 - D - 3D + 2)y = \frac{3}{4}(e^{z}-5)$$

$$(D^2 - 4D + 2)y = \frac{3}{4}(e^{z}-5)$$

The dimiliarly eqn:

$$D^2 - 4D + 2 = 0$$

$$D = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$y_c = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z}$$

$$y_c = C_1 (5+2x)^{2+\sqrt{2}} + C_2 (5+2x)^{2-\sqrt{2}}$$

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$$y_p = \frac{1}{D^2-4D+2}\,\frac{3}{4}(e^z-5)$$

$$= \frac{3}{4}\left[\frac{1}{1-4+2}e^z - \frac{1}{2}5\right]$$

$$= \frac{3}{4}\left[-e^z - \frac{5}{2}\right] = \frac{3}{4}\left[-(5+2x) - \frac{5}{2}\right]$$

$$= \frac{3}{4}\left[-5-2x-\frac{5}{2}\right]$$

$$= \frac{3}{4}\left[-2x-\frac{15}{2}\right]$$

$$y_p = -\left[\frac{3x}{2} + \frac{45}{8}\right]$$

$$y = y_c + y_p$$

$$= c_1(5+2x)^{2+\sqrt{2}} + c_2(5+2x)^{2-\sqrt{2}} - \left[\frac{3x}{4} + \frac{45}{8}\right]$$

Ques 2 $\quad (2x+1)^2\frac{d^2y}{dx^2} - 8(2x+1)\frac{dy}{dx} + 16y = 24x$

Sol $\quad$ Put $(2x+1) = e^z$.

$z = \log(2x+1)$

and $x = \dfrac{e^z-1}{2}$.

$(2x+1)\dfrac{dy}{dx} = 2Dy \qquad$ where $D = \dfrac{d}{dz}$

$(2x+1)^2\dfrac{d^2y}{dx^2} = 4D(D-1)y$

The DE becomes

$$4(D^2-D)y - 8\times 2Dy + 16y = 24\left(\frac{e^z-1}{2}\right)$$

$$(4D^2 - 4D - 16D + 16)y = \frac{24}{2}(e^x - 1)$$

$$(D^2 - 5D + 4)y = 3(e^x - 1)$$

The auxiliary equation:

$$D^2 - 5D + 4 = 0$$
$$(D-1)(D-4) = 0$$
$$D = 1, 4$$

$$y_c = C_1 e^z + C_2 e^{4z}$$
$$y_c = C_1 (2x+1) + C_2 (2x+1)^4$$

$$y_p = \frac{1}{D^2 - 5D + 4} 3(e^z - 1)$$

$$= 3\left[e^z \frac{1}{(D+1)^2 - 5(D+1) + 4} - \frac{1}{D^2 - 5D + 4}\right]$$

$$= 3\left[e^z \frac{1}{D^2 + 2D + 1 - 5D - 5 + 4} - \frac{1}{D^2 - 5D + 4}\right]$$

$$= 3\left[e^z \frac{1}{D^2 - 3D} - \frac{1}{D^2 - 5D + 4}\right]$$

$$= \frac{3}{-3}\left[e^z \frac{1}{\left[D - \frac{D^2}{3}\right]} - \frac{3}{4}\right]$$

$$= -\left[e^z \left[D - \frac{D^2}{3}\right]^{-1} z - \frac{3}{4}\right]$$

$$= -\left[e^z z - \frac{3}{4}\right]$$

$$y_p = -(2x+1)\log(2x+1) - \frac{3}{4}$$

$$y = y_c + y_p$$
$$C_1(2x+1) + C_2(2x+1)^4 - (2x+1)\log(2x+1) - \frac{3}{4}$$

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Variation of Parameters

Ques $(D^2-4D+4)y = e^{2x}\sec^2 x$

The Auxiliary eqn:

$$D^2-4D+4=0$$
$$(D-2)^2=0$$
$$D-2=0$$
$$D=2,2$$

$$y_c = (C_1+C_2x)e^{2x}$$
$$y_c = e^{2x}C_1 + e^{2x}xC_2 = C_1e^{2x}+C_2xe^{2x}$$

Let $y_p = uy_1 + vy_2$

$$y_1 = e^{2x},\ y_2 = xe^{2x}$$

Wronskein $= W = \begin{vmatrix} e^{2x} & xe^{2x} \\ e^{2x} & e^{2x}+2e^{2x} \end{vmatrix} \Leftarrow \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= e^{4x}$$

$$\therefore u = -\int \frac{y_2 Q}{W}dx = -\int \frac{x\cancel{e^{2x}}\cdot \cancel{e^{2x}}\sec^2 x}{\cancel{e^{4x}}}dx$$

$$= -\int x\sec^2 x\,dx = -[x\tan x - \log|\sec x|]$$

$$= -x\tan x + \log|\sec x|$$

$$v = \int \frac{y_1 Q}{W}dx = \int \frac{\cancel{e^{2x}}e^{2x}\sec^2 x}{\cancel{e^{4x}}}dx$$

$$v = \int \sec^2 x\,dx = \tan x$$

$$y_p = uy_1 + vy_2 = [-x\tan x + \log|\sec x|]e^{2x} + \tan x(xe^{2x})$$

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Ques 4 $(D^2+3D+2)y = e^{e^x}$

The auxiliary eqn:

$$D^2+3D+2=0$$
$$(D+1)(D+2)=0$$
$$D=-1,-2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Let $y_p = u y_1 + v y_2$
$$y_1 = e^{-x}, \; y_2 = e^{-2x}$$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$W = -e^{-3x}$$

$$\therefore u = -\int \frac{y_2 Q\,dx}{W} = -\int \frac{e^{-x} \times \cancel{e^{-x}} e^{e^x}\,dx}{-e^{-3x}}$$

Let $e^x = t$
$$e^x dx = dt$$
$$= -\int \frac{\cancel{t}\,\cancel{e^{t}}\,e^t\,dt}{+e^{-3t}}$$
$$= -\int \frac{t\,e^t}{e^{-3t}}\,dt = e^t$$

$$u = e^{e^x}$$

$$v = \int \frac{y_1 Q\,dx}{W} = \int \frac{e^{-2x} e^{e^x}\,dx}{e^{-3x}}$$

$e^x = t$
$$= \int \frac{e^t}{e^{-t}}\,dt$$

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$v = -e^{e^x}(e^x - 1)$

$y_p = e^{e^x}(e^x) + e^{e^x}(e^x - 1)\cdot e^{-2x}$

$y_p = e^{e^x} e^{-2x}$

Ques 5. $(D^2 + a^2)y = \sec(ax)$

The auxiliary eqn:

$D^2 + a^2 + 2aD - 2aD = 0$

$(D+a)^2 - (\sqrt{2aD})^2 = 0$

$D^2 = -a^2$

$D = \pm ai$

$y_c = e^{0a}[c_1 \cos ax + c_2 \sin ax]$

$y_c = e^{0a} c_1 \cos ax + e^{0a} c_2 \sin ax$

$y_p = uy_1 + vy_2$

$y_1 = \cos ax \;,\; y_2 = \sin ax$

$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$a\cos^2 ax + a\sin^2 ax = a$

$\therefore u = -\int \frac{y_2 Q\, dx}{W} = \frac{\sin ax . \sec(ax)\, dx}{a}$

$= -\int \frac{\tan ax}{a} dx = \frac{-1}{a} \frac{\log(\sec ax)}{a}$

$u = \frac{-1}{a^2} \log|\sec ax|$

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$$v = \int \frac{y_1 Q\, dx}{W} = \int \frac{\cos ax \sec ax}{a} dx$$

$$= \int \frac{1}{a} dx = \frac{1}{a}\int 1\, dx$$

$$v = \frac{x}{a}$$

$$y_p = u y_1 + v y_2$$

$$= \frac{-1}{a^2} \log(\sec ax)(\cos ax) + \frac{x}{a}(\sin ax)$$

Date 5/12/19

Homework

Ques 17 $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sec^2 x(1+2\tan x)$

$$A.E = D^2 + 5D + 6 = 0$$

$$D = -2, -3$$

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

$$y_p = u y_1 + v y_2$$

Here, $y_1 = e^{-2x}$, $y_2 = e^{-3x}$

$$W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x}$$

$$\therefore u = -\int \frac{y_2 Q}{W} dx = -\int \frac{e^{-3x} e^{-2x} \sec^2 x(1+2\tan x)}{\cancel{e^{-5x}}}$$

$$= -\int (1+2\tan x)\sec^2 x\, dx = \frac{1}{4}[1+2\tan x]^2$$

$$v = \int \frac{y_1 Q\, dx}{W} = \int \frac{e^{-2x} e^{-2x} \sec^2 x(1+\tan x \times 2)}{e^{-5x}}$$

$$\int e^{x} \sec^2 x(1+2\tan x)\, dx$$

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$$v = -e^{x}\sec^2 x$$

$$\therefore \quad y_p = \frac{1}{4}(1+2\tan x)^2 e^{-2x} - e^{x}\sec^2 x\, e^{-3x}$$

Ques 2 $\quad (D^2+1)y = \operatorname{cosec} x \cot x$

The Auxiliary eqn:

$$D^2+1=0$$
$$D^2=-1$$
$$D=\pm i$$

$$y_c = C_1\cos x + C_2\sin x$$

$$y_p = uy_1 + vy_2$$

Here $y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$
$$=1$$

$$\therefore u = -\int \frac{y_2 Q}{W}dx = -\int \frac{\sin x \operatorname{cosec} x \cot x}{1} dx$$

$$u = -\int \cot x\, dx = -\log|\sin x|$$

$$v = \int \frac{y_1 Q}{W} dx = \int \frac{\cos x \operatorname{cosec} x \cot x}{1} dx$$

$$= \int \cot^2 x\, dx = \int (\operatorname{cosec}^2 x - 1)dx$$

$$= -\cot x - x$$

$$y_p = -\cos x \log(\sin x) - x\sin x - \cos x$$

Numerical Solution Of D.E

⇓

Approximate Solution of D.E

Consider, $\frac{dy}{dx} = \sin x$

$$\int dy = \int \sin x \cdot dx$$

$$y = -\cos x + c \quad \ldots\ldots \text{ G.S.}$$

Suppose initial Cond^n are given i.e. $x_0 = 0, y_0 = 0$

$$\therefore \quad 0 = -\cos(0) + c$$

$$c = 1$$

Now: $y = -\cos x + 1$ .... (Particular Solution)

Here, we have 4 methods:

1. Taylor's Series Method:

Given: $\frac{dy}{dx} = g(x, y)$

Initial Cond^n: $(x_0, y_0) = (0, 0)$

To Find: At $x_1 = x_0 + h$
$[y_1 = ?]$

h → step size, Interval, Increment in x

$$\therefore \boxed{h = x_1 - x_0}$$

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Methodology :- The solution of given D.E looks like

$$y = f(x)$$

$$y_1 = f(x_1) = f(x_0+h)$$

$$= f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

$$\# \left[y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \right] \#$$

Date 25/02/19

Euler's Method :- Given :- $\frac{dy}{dx} = f \cancel{(y,x)} (x,y)$

Initial value :- x_0, y_0

To Find :- at x_n = a given value

$y_n = ?$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Modified Euler's Method :- Given :- $\frac{dy}{dx} = f(x, y)$

Initial value : x_0, y_0

To Find : $x_1 = x_0 + h$

$y_1 = ?$

Method :-

$$y_1 = y_0 + h f(x_0, y_0)$$

First approx to y_1

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1) \right]$$

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Check if $y_1^{(1)} = y_1$ ..... STOP

Else

$\underline{II^{nd}}$ approx to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

Check if $y_1^{(2)} = y_1^{(1)}$ ... STOP

Else,

Third Approx to y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Continue until

$y_1^{(n)} = y_1^{(n-1)}$ till 4 decimal places.

R.K. Method:

Runge-Kutta's Method of 4th order:

Method: We first find 4 constants k_1, k_2, k_3 and k_4 as follows ....

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$K = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$\# \left[y_1 = y_0 + K \right] \#$$

Dt 21/02/19

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6 Marks

M-15 Compute approximate value of y at $x = 0.2$. Given $\frac{dy}{dx} = 1+y^2$ with initial condⁿ

$x_0 = 0$, $y_0 = 0$. Compare it with exact solution.

Sol. Given: $\frac{dy}{dx} = 1+y^2$

Initial condⁿ: $x_0 = 0$, $y_0 = 0$

To Find: At $x_1 = 0.2$, $y_1 = ?$

$\therefore h = x_1 - x_0 = 0.2 - 0$

$h = 0.2$

<u>Methodology:</u>

$y' = 1+y^2$	$y_0' = 1+y_0^2 = 1+0 = 1$
$y'' = 2yy'$	$y_0'' = 2y_0y_0' = 0$
$y''' = 2[yy'' + (y')^2]$	$y_0''' = 2[y_0y_0'' + [y_0']^2]$
	$= 2[0+1] = 2$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0'''$$

$$= 0 + 0.2\times 1 + \frac{0.2^2}{2}\times 0 + \frac{(0.2)^3}{6}\times 2$$

$$= 0.2 + \frac{0.008}{6}\times 2$$

$$= 0.2 + 0.002666$$

$$y_1 = 0.20266$$

<u>Exact solution:</u>

$$\frac{dy}{dx} = 1+y^2$$

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$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + c$$

At $x_0 = 0$, $y_0 = 0$

$$\tan^{-1}(0) = 0 + c$$

$$c = 0$$

$$\therefore \tan^{-1} y = x$$

$$y = \tan x$$

At $x = 0.2$

$$y = \tan(0.2)$$

$$y = 0.20271$$

Ques: Find the values of y at $x = 0.1$ and $x = 0.2$ to 5 decimal places.

Given: $\frac{dy}{dx} = x^2 y - 1$ and $y(0) = 1$

Soln: Given: $\frac{dy}{dx} = x^2 y - 1$

Initial condⁿ: $y(0) = 1$
$[x_0 = 0, y_0 = 1]$

To Find: At
$x_1 = 0.1$, $y_1 = ?$
$x_2 = 0.2$, $y_2 = ?$

Sol. Here, $\boxed{h = 0.1}$

Part I:- $y' = x^2 y - 1$ | $y_0' = x_0^2 y_0 - 1 = -1$

$y'' = 2xy + x^2 y'$ | $y_0'' = 0 + 0 = 0$

$y''' = 2[y + xy'] + 2xy' + x^2 y''$

$= 2y + 4xy' + x^2 y''$ | $y_0''' = 2$

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$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 1 + 0.1 \times (-1) + \frac{(0.1)^2}{2} \times 0 + \frac{(0.1)^3}{6\,3} \times \cancel{2}$$

$$= 1 - 0.1 + 0 + \frac{0.001}{3}$$

$$= 0.9 + 0.00033$$

$$y_1 = 0.90033$$

Part II: Now, we take the initial condⁿ

$$x_1 = 0.1 \;,\; y_1 = 0.90033$$

$$y' = x^2 y - 1$$

$$y'' = 2xy + x^2 y'$$

$$y''' = 2y + 4xy' + x^2 y''$$

$$y_1' = x_1^2 y_1 - 1 = -0.9909$$

$$y_1'' = 2x_1 y_1 + x_1^2 y_1' = 0.1701$$

$$y''' = 2y_1 + 4x_1 y_1' + x_1^2 y''$$

$$= 2.1987$$

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$$

$$= +0.90033 - 0.9909 \times 0.1 + \frac{0.1^2}{2} \times 0.1701$$

$$+ \frac{(0.1)^3}{6} \times 2.1987$$

$$y_2 = 0.8024$$

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Euler's Method:

Ques. Using Euler's method, find the approximate value of y when $x = 1.5$. Taking $h = 0.1$. Given:

$$y(1) = 2$$

$$x_0 = 1, \quad y_0 = 2$$

$$\frac{dy}{dx} = \frac{y-x}{\sqrt{xy}}$$

To Find :- At $x_n = 1.5$

$$y_n = ?$$

$$h = 0.1$$

Sol.

$$f(x,y) = \frac{y-x}{\sqrt{xy}}$$

x_n	y_n	$f(x,y) = y-x/\sqrt{xy}$	$y_{n+1} = y_n + hf(x_n, y_n)$
$x_0 = 1$	$y_0 = 2$	0.7071	$y_1 = y_0 + hf(x_0, y_0) = 2.0707$
$x_1 = 1.1$	$y_1 = 2.0707$	0.6432	$y_2 = y_1 + hf(x_1, y_1) = 2.1350$
$x_2 = 1.2$	$y_2 = 2.1350$	0.5842	$y_3 = 2.1934$
$x_3 = 1.3$	$y_3 = 2.1934$	0.5291	$y_4 = 2.2463$
$x_4 = 1.4$	$y_4 = 2.2463$	0.4772	$y_5 = 2.2940$
$x_5 = 1.5$	$y_5 = 2.2940$		

$$\therefore \begin{bmatrix} x_5 = 1.5 \\ y_5 = 2.2940 \end{bmatrix} \text{ Ans}$$

Modified Euler's Method

Que:- Solve $\frac{dy}{dx} = 2 + \sqrt{xy}$.

when $x_0 = 1.2$ and $y_0 = 1.6403$

Find y for $x = 1.4$ with $h = 0.2$.

Sol:

$f(x_0, y_0) = 2 + \sqrt{xy}$

Given: $\frac{dy}{dx} = 2 + \sqrt{xy}$

$\therefore f(x, y) = 2 + \sqrt{xy}$

To Find:- $x_1 = x_0 + h$

$h = 0.2$

$y_1 = ?$

Sol.

$y_1 = y_0 + h f(x_0, y_0)$

$= 1.6403 + 0.2 (1.2, 1.6403)$

$= 1.6403 + 0.2 (3.4030)$

$y_1 = 2.3209$

First approx to y_1

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$

$= 1.6403 + \frac{0.2}{2}\left(3.4030 + (1.4, 2.3209)\right)$

$= 1.6403 + \frac{0.2}{2}(3.4030 + 3.80257)$

$= 1.6403 + 0.72055$

$y_1^{(1)} = 2.3608$

II approx to y_1:

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + (f\ x_1, y_1')\right.$

$$y_1^{(2)} = 1.6403 + \frac{0.2}{2}\left[3.4030 + f(1.6, 2.3608)\right]$$

$$= 1.6403 + \frac{0.2}{2}\left[3.4030 + 3.9435\right]$$

$$= 1.6403 + 0.7347$$

$$y_1^{(2)} = 2.3624$$

IIIrd approx to y_1:

$$y_1^{(3)} = 1.6403 + \frac{0.2}{2}\left[3.4030 + f(1.8, 2.3624)\right]$$

$$= 2.3625$$

IVth approx to y_1:

$$y_1^{(4)} = 1.6403 + \frac{0.2}{2}\left[3.4030 + f(2, 2.3624)\right]$$

$$y_1^{(4)} = 2.3625$$

$$y_1^{(4)} = y_1^{(3)} = 2.3625$$

Ques. Using modified Euler's method find approximate value of y at $x = 0.2$ by taking $h = 0.1$ and using 3 iteration.

Given: $\frac{dy}{dx} = x + 3y$

Initial value: $y = 1$ and $x = 0$

Sol. Given: $\frac{dy}{dx} = x + 3y$

$$\therefore f(x, y) = x + 3y.$$

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To Find: $x_1 = x_0 + h$

$h = 0.1$

$y = ?$

Part I:

Sol.

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.1 f(0, 3y_1)$

$= 1 + 0.1 \times 3$

$y_1 = 1.3000$

I^{st} approx to y_1

$y_1^{(1)} = y_0 + \frac{h}{2}[f(f(x_0, y_0) + f(x_1, y_1')]$

$= 1 + \frac{0.1}{2}[1.3 + (0.1, 1.300)]$

$= 1 + 0.3500$

$= \cancel{1.3150}\ \cancel{1.3583}\ 1.3500$

II^{nd} approx to y_1

$y_1^{(2)} = y_0 + \frac{h}{2}[1.3 + f(0.1, 1.3150)$

$= 1.3583$

Third approx to y_1

$y_1^{(3)} = y_0 + \frac{h}{2}[1.3 + f(a_1, y_1^{(2)})]$

$y_1^{(3)} = 1.3586$

Part II: Now, taking $x_1 = 0.1$, $y_1 = 1.3586$ as initial value

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.7762$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

$$= 1.3586 + \frac{0.1}{2}\left[\cancel{5.1758}^{4.17} + 5.5286\right]$$

$$= \cancel{1.8938}\ 1.8438$$

$$y_2^{(2)} = 1.3586 + \frac{0.1}{2}\left[4.1758 + 5.8314\right]$$

$$y_2^{(2)} = 1.8540$$

$$y_2^{(3)} = 1.8555$$

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Ques. $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$. Find y when x is 0.5 where $h = 0.5$.

Given : $\frac{dy}{dx} = \frac{1}{x+y}$

$$f(x,y) = \frac{1}{x+y}$$

Initial condition :- $y(0) = 1$

i.e. $x_0 = 0$, $y_0 = 1$

$h = 0.5$

To Find: $x_1 = 0.5$, $y_1 = ?$

Sol. $y_1 = y_0 + h f(x_0, y_0)$

$$K_1 = hf(x_0, y_0)$$
$$= 0.5 f(0,1)$$
$$= 0.5$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= 0.5\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$

$$= 0.5(0.2500, 1.2500)$$

$$K_2 = 0.3333$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.3529.$$

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$$k_4 = hf\left(x_0 + h, \; y_0 + k_3\right)$$

$$= 0.5\left(0 + 0.5, \; 1 + 0.3529\right)$$

$$k_4 = 0.2698$$

$$K = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$K = 0.3570$$

$$y_1 = y_0 + K$$

$$= 1 + 0.3570$$

$$\boxed{y_1 = 1.3570}$$

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Ques.

Given: $\frac{du}{dt} = -2tu^2$

$f(t,u) = -2tu^2$

Initial condⁿ: $u(0) = 1$

i.e. $t_0 = 0$, $u_0 = 1$

$h = 0.2$

Sol. $t_0 = 0$, $u_0 = 1$, $h = 0.2$, $t = 0.2$

$k_1 = hf(t_0, y_0) = 0.2[-2\times0\times1^2]$
$= 0$

$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right)$

$= -0.2[2\times0.1\times1^2]$
$= -0.04$

$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right)$

$= -0.0384$

$k_4 = hf(t_0 + h, u_0 + k_3)$
$= -0.0740$

$K = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = -0.0385$

$u = u_0 + K = 1 - 0.0385$

$u = 0.9615$

Betta and Gamma Functions

Gamma Function: If $n \in Q^+$ (+ve rational) then Gamma of n is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Properties of Γn :-

(i) $\Gamma 1 = 1$

Proof:- $\Gamma 1 = \int_0^\infty e^{-x} dx = [-e^{-x}]_0^\infty = 0+1$

$= 1$

(ii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(iii) $\Gamma n+1 = n \Gamma n$ or $\Gamma n+1 = n!$

By definition:- $\Gamma n+1 = \int_0^\infty e^{-x} x^{n+1-1} dx$

$$= \left[x^n \int e^{-x} dx\right]_0^\infty - \int_0^\infty \left[\int e^{-x} dx \frac{d}{dx}(x^n)\right] dx$$

$$= \left[\frac{x^n e^{-x}}{(-1)}\right]_0^\infty - \int_0^\infty -e^{-x} n x^{n-1} dx$$

$$= 0 + n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma n+1 = n \Gamma n$$

Ques. $\Gamma 5 = 4\Gamma 4 = 4\cdot 3\Gamma 3$

$= 4\cdot 3\cdot 2\Gamma 2$

$= 4\cdot 3\cdot 2\cdot 1\Gamma 1$

$= 4\cdot 3\cdot 2\cdot 1$

$\Gamma 5 = 4!$

lly, $\Gamma 10 = 9!$

Ques. $\Gamma\frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$

$= \frac{5}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$= \frac{5}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

(iv) $\Gamma n\Gamma 1-n = \frac{\pi}{\sin n\pi}$

Example: $\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \frac{\pi}{\sin(\pi/4)} = \sqrt{2}\,\pi$

(v) $\int_0^\infty e^{-\cancel{b}x} x^{n-1}\,dx = \frac{\Gamma n}{b^n}$

Beta Function:- If $m, n \in Q^+$ (+ve rational numbers)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of $\beta(m,n)$:-

① $\beta(m,n) = \beta(n,m)$

② $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

③ $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

x	0	1
θ	0	$\pi/2$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

④ In property ③, put

$2m-1 = p, \quad 2n-1 = q$

$\therefore\ m = \dfrac{p+1}{2}, \quad n = \dfrac{q+1}{2}$

$$\#\# \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Type I of Beta Function:-

$$\int_0^a x^{m-1}(a^p - x^p)^{n-1} dx$$

Put $x^p = a^p t$

$x = a t^{1/p}$

$dx = \frac{1}{p} a t^{1/p} dt$

Ex: $I = \int_0^2 x^4 (8 - x^3)^{-1/3} dx$

x	0	2
t	0	1

Put $x^3 = 8t$

$x = 2t^{1/3}$

$dx = \frac{2}{3} t^{-2/3} dt$

$$I = \int_0^1 (2t^{1/3})^4 (8 - 8t)^{-1/3} \frac{2}{3} t^{-2/3} dt$$

$$= \frac{2^{4-1+1}}{3} \int_0^1 t^{4/3 - 2/3} (1-t)^{-1/3} dt$$

$$= \frac{2^4}{3} \int_0^1 t^{2/3} (1-t)^{-1/3} dt = \frac{2^4}{3} \int_0^1 t^{5/3 - 1} (1-t)^{\frac{2}{3}-1} dt$$

$$= \frac{2^4}{3} \beta\left(\frac{5}{3}, \frac{2}{3}\right)$$

Ex: $I = \int_0^{2a} x^m \sqrt{2ax - x^2}\, dx$

Taking x common

$$= \int_0^{2a} x^m x^{1/2} (2a - x)^{1/2} dx$$

x	0	$2a$
t	0	1

Put $x = 2at$

$$= \int_0^{2a} x^m x^{1/2} \quad dx = 2\cancel{t}\,a dt$$

$$= \int_0^{1} x^m x^{1/2} (2a - 2at)^{1/2} 2a dt$$

$$= \cancel{2a} \int_0^1 x^m x^{1/2} (1-t)^{1/2} 2a dt = 4a^2 \int_0^1 x^m x^{1/2} (1-t)^{1/2} dt$$

$$= \cancel{4a^2} \int_0^1 (2at)^{m+1/2} (1-t)^{1/2} dt$$

$$I = (2a)^{m+\frac{3}{2}-\frac{1}{2}} \int_0^1 t^{m+3/2}(1-t)^{3/2}\, dt$$

$$I = 2a^{m+2} \beta\left(\frac{3}{2}, m+\frac{3}{2}\right)$$

Ques. (3) $I = \int_0^1 \sqrt{1-\sqrt{x}}\, dx \int_0^{1/2} \sqrt{2y-4y^2}\, dy = \frac{\pi}{30}$

Sol.

$$I_1 = \int_0^1 (1-\sqrt{x})^{1/2} dx$$

Put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\, dt$

$$I_1 = \int_0^1 (1-t)^{1/2} 2t\, dt$$

$$= 2\int_0^1 (1-t)^{1/2} t\, dt$$

$$= 2\int_0^1 (1-t)^{\frac{3}{2}-1} t^{2-1} dt$$

$$I_1 = 2\beta\left(2, \frac{3}{2}\right)$$

$$I_2 = \int_0^{1/2} \sqrt{2y-4y^2}\, dy$$

$$I_2 = \int_0^{1/2} (2y)^{1/2}(1-2y)^{1/2} dy$$

Put $2y = t$

$y = t/2$

$dy = dt/2$

y	0	1/2
t	0	1

$$I_2 = \int_0^1 t^{1/2}(1-t)^{1/2} dt$$

$$I_2 = \frac{1}{2} \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$L.H.S. = I_1 \cdot I_2 = \cancel{2}\, \beta\left(2, \frac{3}{2}\right) \cdot \frac{1}{\cancel{2}} \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma 2\, \Gamma 3/2}{\Gamma\left(2+\frac{3}{2}\right)} \cdot \frac{\Gamma 3/2\, \Gamma 3/2}{\Gamma 3}$$

$$I = \frac{1\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^3}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot 2} = \frac{(\sqrt{\pi})^2}{30}$$

$$I = \frac{\pi}{30} \quad \text{Ans}$$

Type II of Beta Function :-

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$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Ques

$$I = \int_0^{\pi/6} \cos^3 3\theta \sin^2 6\theta \, d\theta$$

$$= \int_0^{\pi/6} \cos^3 3\theta \, (2\sin 3\theta \cos 3\theta)^2 \, d\theta$$

$$= 4\int_0^{\pi/6} \sin^2 3\theta \cdot \cos^5 3\theta \, d\theta$$

Put $3\theta = t$

$d\theta = dt/3$

x	0	$\pi/6$
t	0	$\pi/2$

$$I = 4\int_0^{\pi/2} \sin^2 t \cos^5 t \, \frac{dt}{3}$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t \cos^5 t \, dt$$

$$= \frac{4}{3} \cdot \frac{1}{2} \beta\left(\frac{3}{2}, \frac{6}{2}\right)$$

$$I = \frac{2}{3} \beta\left(\frac{3}{2}, 3\right)$$

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Ques (2) $$I = \int_{-\pi}^{\pi} \sin^2 x \cos^4 x \, dx$$

f (x) is even fn.

$\because f(-x) = f(x)$

$$I = 2\int_{0}^{\pi} \sin^2 x \cos^4 x \, dx$$

$$I = 2\int_{0}^{\pi/2} \left[\sin^2 x \cos^4 x + \sin^2(\pi - x) \cos^4(\pi - x)\right] dx$$

$$= 2\int_{0}^{\pi/2} \left[\sin^2 x \cos^4 x + \cos^4 x \sin^2 x\right] dx$$

$$= 4\int_{0}^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$= 4\left[\frac{1}{2} \beta\left(\frac{3}{2}, \frac{5}{2}\right)\right]$$

$$I = 2 \beta\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$I_1 = \int_{0}^{\infty} x e^{-x^2} dx$$

Put $x^2 = t \Rightarrow x = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} dt$

x	0	∞
t	0	∞

$$\therefore I_1 = \int_{0}^{\infty} t^{1/2} e^{-t} \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-t} t^{-1/2 + 1/2} dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-t} dt$$

$$I_2 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$$

Put $x^2 = t$

$$dx = \frac{1}{2} t^{-1/2} dt$$

x	0	∞
t	0	∞

$$I_2 = \int_0^\infty e^{-t} \left(t^{1/2}\right)^{-1/2} \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} t^{-3/4} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} t^{\frac{1}{4}-1} dt$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right)$$

Gamma Function:

Type I :-

$$\int x^m e^{-x^n} dx$$

$$\boxed{\begin{aligned} \text{Put } x^n &= t \\ x &= t^{1/n} \end{aligned}}$$

Ques 1

$$I = \int_0^\infty \sqrt{x} \; e^{-\sqrt[3]{x}} dx$$

Put $\sqrt[3]{x} = t$

$x = t^3$

$dx = 3t^2 dt$

x	0	∞
t	0	∞

$$I = \int_0^\infty \sqrt{t^3} e^{-t} 3t^2 dt$$

$$= 3\int_0^\infty e^{-t} t^{3/2+2} dt$$

$$= 3\int_0^\infty e^{-t} t^{7/2} dt$$

$$= 3\int_0^\infty e^{-t} t^{9/2-1} dt$$

$$= 3\sqrt{\tfrac{9}{2}}$$

$$= 3 \cdot \frac{7}{2} \cdot \frac{5}{3} \cdot \frac{1}{2} \cdot \frac{3}{2}\sqrt{\tfrac{1}{2}}$$

$$= \frac{315}{16}\sqrt{\pi}$$

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Que ⑤ $I = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx \cdot \int_0^\infty y^4 \cdot e^{-y^6} dy = \frac{\pi}{9}$

Sol. Let $I = I_1 \cdot I_2$

Consider $I_1 = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx$

Put $x^3 = t$

$x = t^{1/3}$

$dx = \frac{1}{3} t^{-2/3} dt$

x	0	∞
t	0	∞

$I_1 = \int_0^\infty \frac{e^{-t}}{\sqrt{t^{1/3}}} \frac{1}{3} t^{-2/3} dt$

$I_1 = \int_0^\infty e^{-t} \frac{1}{3} t^{-2/3 - \frac{1}{6}} dt$

$I_1 = \frac{1}{3} \int_0^\infty e^{-t} t^{-5/6} dt$

$I_1 = \frac{1}{3} \int_0^\infty e^{-t} t^{\frac{1}{6}-1} dt$

$I_1 = \frac{1}{3} \Gamma \frac{1}{6}$

Now $I_2 = \int_0^\infty y^4 \cdot e^{-y^6} dy$

Put $y^6 = t$

$y = t^{1/6}$

$dy = \frac{1}{6} t^{-5/6} dt$

$I_2 = \int_0^\infty y^4 \cdot e^{-t} dy$ ~~$= \int_0^\infty y^4 \cdot e^{-t}$~~

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$$I_2 = \int_0^\infty (t^{1/6})^4 \, e^{-t} \, \frac{1}{6} \, t^{-5/6} \, dt$$

$$I_2 = \frac{1}{6} \int_0^\infty e^{-t} \, t^{\frac{5}{6}-1} \, dt$$

$$I_2 = \frac{1}{6} \Gamma\left(\frac{5}{6}\right)$$

Now

$$\begin{aligned} I &= I_1 \cdot I_2 \\ &= \frac{1}{3} \Gamma\left(\frac{1}{6}\right) \cdot \frac{1}{6} \Gamma\left(\frac{5}{6}\right) \\ &= \frac{1}{18} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right) \\ &= \frac{1}{18} \cdot \frac{\pi}{\sin(\pi/6)} \end{aligned}$$

$$I = \frac{1}{\cancel{18}\,9} \cdot \cancel{2}\pi$$

$$I = \frac{\pi}{9}$$

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Type - II :-

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$$

Put $\log \frac{1}{x} = t$

$$\frac{1}{x} = e^t$$

$$x = e^{-t}$$

$$\int_0^1 x^m (\log x)^n dx$$

$$\log_e x = -t$$

$$x = e^{-t}$$

$$\boxed{dx = -e^{-t} dt}$$

Ques ③ $I = \int_0^1 (x \log x)^4 dx$

Sol. Put $\log x = -t$

$$\boxed{x = e^{-t}}$$

$$dx = -e^{-t} dt$$

x	0	1
t	∞	0

$$I = \int_\infty^0 [e^{-t}(-t)]^4 (-e^{-t}) dt$$

$$= -\int_\infty^0 e^{-5t} t^4 dt$$

$$= \int_0^\infty e^{-5t} t^4 dt$$

Using, $\int_0^\infty e^{-bx} x^{n-1} = \frac{\sqrt{n}}{b^n}$

$$I = \int_0^\infty e^{-5t} t^{5-1} dt = \frac{\sqrt{5}}{5^5} = \frac{4!}{5^5}$$

$$I = \frac{24}{5^5}$$

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$\log 0 = -\infty$

$\log 1 = 0$

$\log \infty = \infty$

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Ques 4 $I = \int_0^1 \frac{1}{\sqrt{\log x}}\, dx$

Ques 5 $I = \int_0^1 \sqrt{x} \log\left(\frac{1}{x}\right) dx$

(4) $I = \int_0^1 \frac{1}{\sqrt{-\log x}}\, dx$

Sol. Put $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t}\, dt$

x	0	1
t	∞	0

$\therefore \cancel{I = -\int_\infty^0 e^{-t}}$

$\therefore I = \int_\infty^0 t^{-1/2} (-e^{-t}) dt$

$= -\int_\infty^0 t^{-1/2} e^{-t} dt$

$I = \int_0^\infty t^{-1/2} e^{-t} dt$ ~~using~~ $\cancel{\int_0^\infty e^{-x}}$

$I = \int_0^\infty e^{-t} t^{1/2} dt$

$I = \Gamma\frac{1}{2}$

$I = \sqrt{\pi}$

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Ques 5) $I = \int_0^1 \sqrt{x \log\left(\frac{1}{x}\right)}\, dx$

Put $\log \frac{1}{x} = t$

$\frac{1}{x} = e^{t}$

$x = e^{-t}$

$dx = -e^{-t}\, dt$

x	0	1
t	∞	0

$\therefore I = \int_\infty^0 \sqrt{e^{-t} t}\,(-e^{-t})\, dt$

$= -\int_\infty^0 \sqrt{t}\; e^{-2t}\, dt$

$= \int_0^\infty \sqrt{t'}\, e^{-2t}\, dt$

$= \int_0^\infty t^{1/2} (e^{-2t})^{1/2}\, dt$

$= \int_0^\infty t^{1/2}\, e^{-t}\, dt$

$I = \sqrt{\pi}$

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Type-III :-

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$$\int_0^\infty x^m a^{-x}\,dx$$

Put $a^{-x} = e^{-t}$

$-x\log a = -t$

$$\boxed{x = \frac{+t}{\log a}}$$

Eg 1 $\quad I = \displaystyle\int_0^\infty \frac{x^4}{4^x}\,dx = \int_0^\infty x^4 \cdot 4^{-x}\,dx$

Put $4^{-x} = e^{-t}$

$-x\log 4 = -t$

$$\boxed{x = \frac{t}{\log 4}}$$

$$dx = \frac{dt}{\log 4}$$

$$I = \int_0^\infty x^4 \cdot e^{-t} \cdot \frac{dt}{\log 4}$$

$$= \int_0^\infty \left(\frac{t}{\log 4}\right)^4 \cdot e^{-t}\,\frac{dt}{\log 4}$$

$$= \int_0^\infty \frac{t^4 \cdot e^{-t}}{(\log 4)^4}\,\frac{dt}{\log 4} = \int_0^\infty \frac{t^4 \cdot e^{-t}}{(\log 4)^5}\,dt$$

$$= \frac{1}{(\log 4)^5}\int_0^\infty e^{-t}\,t^{5-1}\,dt$$

$$I = \frac{\sqrt{5}}{(\log 4)^5}$$

Ques (7) $I = \int_0^\infty 5^{-4x^2} dx$

Put $5^{-4x^2} = e^{-t}$

$\cancel{-} 4x^2 \log 5 = \cancel{-} t$

$4x^2 \log 5 = t$

$x^2 = \dfrac{t}{4\log 5}$

$x = \dfrac{\sqrt{t}}{2\sqrt{\log 5}}$

$dx = \dfrac{1}{2} \dfrac{t^{-1/2}}{2\sqrt{\log 5}} dt$

$dx = \dfrac{1}{4} \dfrac{t^{-1/2}}{\sqrt{\log 5}} dt$

$I = \int_0^\infty e^{-t} \dfrac{1}{4} \dfrac{t^{-1/2}}{\sqrt{\log 5}} dt$

$I = \left[\dfrac{1}{4\sqrt{\log 5}}\right] \int_0^\infty e^{-t} t^{-1/2} dt$

$I = \dfrac{1}{4\sqrt{\log 5}} \sqrt{\tfrac{1}{2}}$

$I = \dfrac{\sqrt{\pi}}{4\sqrt{\log 5}}$

H/W

Ques (8) $I = \int_0^\infty x^2 7^{-4x^2} dx$

Put $7^{-4x^2} = e^{-t}$

$\cancel{-} 4x^2 \log 7 = \cancel{-} t$

$4x^2 \log 7 = t$

$x^2 = \dfrac{t}{4\log 7}$

$x = \dfrac{\sqrt{t}}{2\sqrt{\log 7}}$

Type-IV :-

11/12 (9) $\int_0^\infty x^n \cos mx\,dx$ = Real part of $\int_0^\infty x^n e^{-imx}\,dx$

$\int_0^\infty x^n \sin mx\,dx$ = (-) Img part of $\int_0^\infty x^n e^{-imx}\,dx$

(1) Let $I_1 = \int_0^\infty x^n e^{-imx}\,dx$

Put $imx = t$

Hint :- $\left(i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \right)$

$dx = \frac{1}{4}\frac{t^{-1/2}}{\sqrt{\log 7}}\,dt$

$\therefore I = \int_0^\infty e^{-t}\frac{t}{4\log 7} \cdot \frac{1}{4}\frac{t^{-1/2}\,dt}{\sqrt{\log 7}}$

$= \frac{1}{16\sqrt{\log 7}}\int_0^\infty e^{-t} t^{-1/2}\,dt$

$= \frac{1}{16\sqrt{\log 7}}\int_0^\infty e^{-t} t^{-1/2}\,dt$

$I = \frac{1}{16\sqrt{\log 7}}\int_0^\infty e^{-t} t^{3/2}\,dt$

$I = \frac{1}{16\sqrt{\log 7}}\overline{\left|\frac{3}{2}\right.}$

$= \frac{1}{16\sqrt{\log 7}}\frac{1}{2}\overline{\left|\frac{1}{2}\right.}$

$I = \frac{1}{32\sqrt{\log 7}}\sqrt{\pi}$

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Type-IV :

Ques (1) $\int_0^\infty x^{m-1} \cos ax \, dx$

Sol

$$I = \text{Real part of } \int_0^\infty x^{m-1} e^{-iax} dx$$

$$\text{Let } I_1 = \int_0^\infty x^{m-1} e^{-iax} dx$$

Put $iax = t$

$$x = \frac{t}{ia}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) ax = t$$

$$ia\,dx = dt$$

$$dx = \frac{dt}{ia}$$

$$I_1 = \int_0^\infty a^{m-1} e^{-t} \frac{dt}{ia}$$

$$= \int_0^\infty \left(\frac{t}{ia}\right)^{m-1} e^{-t} \frac{dt}{ia} = \frac{1}{(ia)^m} \int_0^\infty e^{-t} t^{m-1} dt$$

$$I_1 = \frac{\Gamma m}{a^m i^m} = \frac{\Gamma m}{a^m} i^{-m}$$

$$I_1 = \frac{\Gamma m}{a^m}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{-m}$$

$$I_1 = \frac{\Gamma m}{a^m}\left[\cos\frac{m\pi}{2} - i\sin\left(\frac{m\pi}{2}\right)\right]$$

$$I = \text{Real part of } I_1 = \frac{\Gamma m}{a^m}\cos\left(\frac{m\pi}{2}\right) \quad \underline{\text{Ans}}$$

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Basics of Integration:-

① $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

② $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

③ $$\int \frac{1}{a+b\cos^2 x}\,dx$$

④ $$\int \frac{1}{a+b\sin^2 x}\,dx$$

Divide Nr and Dr by $\cos^2 x$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

⑤ $$\int \frac{1}{a\cos^2 x + b\sin^2 x}\,dx$$

⑥ $$\int \frac{1}{a+b\cos x}$$

⑦ $$\int \frac{1}{a+b\sin x}$$

⑧ $$\int \frac{1}{a\cos x + b\sin x + c}$$

Put $\tan(x/2) = t$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

{The final answer will be in terms of $\tan^{-1}$ for (3, 4, 5, 6, 7, 8)}

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$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\,dx$

Put $x-a = t^2$

$\int_a^b x^n (x-c)^{p/q}\,dx$

Put $x-c = t$

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Properties of Beta :-

Property 5 :-

(i) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Proof :-

$$I = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Put $x = \tan^2\theta$

x	0	∞
θ	0	$\pi/2$

$dx = 2\tan\theta \sec^2\theta \, d\theta$

$$I = \int \frac{\tan^{2m-2}\theta}{\sec^{2n+2m}} 2\tan\theta \sec^2\theta \, d\theta$$

Convert everything to $\sin\theta$, $\cos\theta$

$$I = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$I = \beta(m,n)$ ..... Property No. 3

(ii) Prove that :-

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} = \frac{\beta(m,n)}{a^n b^m}$$

$$I = \frac{1}{a^{m+n}} \int_0^\infty \frac{x^{m-1}}{\left(1+\frac{bx}{a}\right)^{m+n}} dx$$

Put $bx/a = \tan^2\theta$

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Put $bx = ay$

$$\therefore x = \frac{a}{b} y$$

$$dx = \frac{a}{b} dy$$

x	0	∞
θ	0	∞

$$I = \int \frac{\frac{a^{m-1} y^{m-1}}{b^{m-1}}}{(a+ay)^{m+n}} * \frac{a}{b} \cdot dy$$

$$= \frac{a^m}{b^m} \int \frac{y^{m-1}}{a^{m+n} (1+y)^{m+n}} dy$$

$$= \frac{a^m}{a^{m+n} b^m} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$I = \frac{\beta(m, n)}{a^n \cdot b^m}$$

Property 6 :-

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Duplication Formula :-

$$\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}\, \Gamma 2m}{2^{2m-1}}$$

Proof :-

$$\frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

Put $q = p$

$$\frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^p\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta \cos\theta)^p \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^p d\theta$$

$$= \frac{1}{2^p} \int_0^{\pi/2} \sin^p 2\theta \, d\theta$$

Put $2\theta = t$

$d\theta = dt/2$

$$= \frac{1}{2^p} \int_0^{\pi} \sin^p t \, \frac{dt}{2}$$

$$= \frac{1}{2^p \cdot 2} \int_0^{\pi/2} \left[\sin^p t + \sin^p(\pi - t) \right] dt$$

$$= \frac{1}{2^p \cdot 2} \int_0^{\pi/2} (2\sin^p t)\, dt$$

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \frac{1}{2^p}\left[\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)\right]$$

Put $\frac{p+1}{2} = m \Rightarrow p = 2m-1$

$$\beta(m, m) = \frac{1}{2^p}\beta\left(m, \frac{1}{2}\right)$$

$$\frac{\Gamma m \cdot \Gamma m}{\Gamma 2m} = \frac{1}{2^p} \frac{\Gamma m \, \Gamma 1/2}{\Gamma m+1/2}$$

$$\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi} \, \Gamma 2m}{2^{2m-1}}$$

Ques 1 $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}$

Sol. Put $x^2 = t$

$x = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} dt$

x	0	1
t	0	1

$$I = \int_0^1 \frac{t^n}{\sqrt{1-t}} \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^1 \frac{t^n \, t^{-1/2}}{(1-t)^{1/2}} dt$$

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$$= \frac{1}{2}\int_0^1 t^n t^{-1/2}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{n-1/2}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{2}\,\beta\left(n+1/2,\ 1/2\right)$$

$$= \frac{1}{2}\,\frac{\Gamma(n+1/2)\,\Gamma(1/2)}{\Gamma(n+1)}$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{1}{(n!)}\,\frac{2^{1-2n}\sqrt{\pi}\,\Gamma(2n)}{\Gamma(n)}$$

$$= \frac{\sqrt{\pi}}{2^{2n}}\,\frac{1}{n!}\,\sqrt{\pi}\,\frac{2n}{2n}\,\frac{\Gamma(2n)}{\Gamma(2n)}$$

$$= \frac{\pi}{(2)^{2n}}\,\frac{1}{n!}\,\frac{\Gamma(2n+1)}{2\,\Gamma(n+1)}$$

$$I = \frac{\pi}{2}\,\frac{1}{(2)^{2n}}\,\frac{(2n)!}{(n!)^2}$$

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Type III of Beta :-

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Ques 1

$$I = \int_0^\infty \frac{\sqrt{x}}{(1+2x+x^2)}\,dx$$

$$= \int_0^\infty \frac{x^{1/2}}{(1+x)^2}\,dx$$

$$= \int_0^\infty \frac{x^{\frac{3}{2}-1}}{(1+x)^{\frac{3}{2}+\frac{1}{2}}}\,dx$$

$$= \beta\left(\frac{3}{2},\frac{1}{2}\right)$$

$$= \frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} = \frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1}$$

$$I = \frac{\pi}{2}$$

Ques ②

$$I = \int_0^\infty \frac{x^5(1+x^4)}{(1+x)^{16}}\,dx$$

$$= \int_0^\infty \left[\frac{x^5+x^9}{(1+x)^{16}}\right]dx$$

$$= \int_0^\infty \frac{x^5\,dx}{(1+x)^{16}} + \int_0^\infty \frac{x^9}{(1+x)^{16}}\,dx$$

$$= \int_0^\infty \frac{x^{6-1}}{(1+x)^{10+6}}\,dx \,\cancel{dx} + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+6}}\,dx$$

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$$I = \beta(6,10) + \beta(10,6)$$

$$I = 2\,\beta(10,6)$$

Ans ③ $$I = \int_0^\infty \frac{(x^6 - x^3) \cdot x^2\,dx}{(1+x^3)^5}$$

Put $x^3 = t$

$3x^2\,dx = dt$

$x^2\,dx = dt/3$

x	0	∞
t	0	∞

$$I = \int_0^\infty \frac{(t^2 - t)}{(1+t)^5} \cdot dt/3$$

$$= \frac{1}{3}\left[\int_0^\infty \frac{t^2}{(1+t)^5}\,dt - \int_0^\infty \frac{t}{(1+t)^5}\,dt\right]$$

$$= \frac{1}{3}\left[\int \frac{t^{3-1}}{(1+t)^{3+2}}\,dt - \int_0^\infty \frac{t^{2-1}}{(1+t)^{3+2}}\,dt\right]$$

$$= \frac{1}{3}\left[\beta(3,2) - \beta(2,3)\right]$$

$$I = \frac{1}{3}\beta(3,2) - \frac{1}{3}\beta(2,3) \quad \underline{\text{Ans}}$$

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Type IV of Beta:-

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

Ques 1
$$I = \int_1^\infty \frac{x^{n/2-1}\,dx}{(1+x)^n}$$

Put $x = \frac{1}{t}$

$dx = \frac{-1}{t^2}dt$

x	1	∞
t	1	0

$$I = \int_1^0 \frac{(1/t)^{\frac{n}{2}-1}}{(1+1/t)^n}\left(\frac{-1}{t^2}\right)dt$$

$$I = \int_0^1 \frac{t^{\frac{-n+1-2+n}{2}}}{(1+t)^n}\,dt$$

$$I = \int_0^1 \frac{t^{\frac{n}{2}-1}}{(1+t)^n}\,dt.$$

$$\text{N.T.S.C} \Longleftarrow I = \frac{1}{2}\int_0^1 \frac{t^{\frac{n}{2}-1} + t^{n/2-1}}{(1+t)^{n/2+n/2}}\,dt$$

$$I = \frac{1}{2}\,\beta\left(\frac{n}{2},\frac{n}{2}\right).\ \text{Ans}$$

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Ans 2 $I = \int_0^\infty \frac{1}{(e^x+e^{-x})^n} dx = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$

$$I = \int_0^\infty \frac{1}{(e^x + 1/e^x)^n} dx$$

$$= \int_0^\infty \frac{e^{xn}}{(1+e^{2x})^n} dx$$

Put $e^{2x} = t$

$e^{2x}\ t^{1/2}$

$x = \frac{1}{2} \log t$

$dx = \frac{1}{2t} dt$

x	0	∞
t	1	∞

$$I = \int_1^\infty \frac{t^{n/2}}{(1+t)^n 2t} dt$$

$$= \frac{1}{2} \int_1^\infty \frac{t^{n/2-1}}{(1+t)^n} dt$$

$$I = \frac{1}{2} \cdot \frac{1}{2} \int_1^\infty \frac{t^{\frac{n}{2}-1} + t^{n/2-1}}{(1+t)^{n/2+n/2}} dt$$

$$I = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

Hence Proved

Type V of Beta :-

Ques ①

$$\int_a^b (b-x)^n (x-a)^m \, dx$$

Put $(x - LL) = (UL - LL)\, t$

$$(x-a) = (b-a)t$$

$$x = a + (b-a)t$$

$$dx = (b-a)\,dt$$

$$b - x = b - a - (b-a)t$$

$$= (b-a)(1-t)$$

x	a	b
t	0	1

$$I = \int_0^1 [(b-a)(1-t)]^n ((b-a)t)^m (b-a)\,dt$$

$$= \cancel{b-a} \int_0$$

$$I = (b-a)^{m+n+1} \int_0^1 t^{m+1-1} (1-t)^{n+1-1} \, dt$$

$$I = (b-a)^{m+n+1} \beta(m+1, n+1)$$

Ques 2

$$I = \int_3^7 4\sqrt{(7-x)(x-3)} \, dx$$

Put $(x-3) = (7-3).t$

$$x - 3 = 4t$$

$$x = 4t + 3$$

$$dx = 4dt$$

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$$6 - x \Rightarrow 7 - x = 7 - (4t+3)$$
$$= 7 - 4t - 3$$
$$= -4t - 4$$
$$= -4(t+1)$$

x	3	7
t	0	1

$$I = \int_0^1 \sqrt[4]{[4(t+1)]}\,(4t)\; 4dt$$

$$= \cancel{4}\int_0^1 [4(t+1)]^{1/4} (4t)^{1/4}\; 4dt$$

$$= 4^{\frac{1}{4}+\frac{1}{4}+1} \int_0^1 (t+1)^{1/4} (t)^{1/4}\, dt$$

$$= 4^{3/2} \int_0^1 (t+1)^{\frac{5}{4}-1} (t)^{\frac{5}{4}-1}\, dt$$

$$= 4^{3/2}\, \beta\left(\frac{5}{4}, \frac{5}{4}\right)$$

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Differentiation Under Integral Sign (DUIS)

Consider this

$$I = \int_0^1 \frac{(x^\alpha - 1)}{\log x}\, dx$$

This Integral is difficult to solve by usual methods of integration. After evaluation the final sol^n will be fn of α. Hence we denote this integral by $I(\alpha)$.

$$I(\alpha) = \int_0^1 \frac{(x^\alpha - 1)}{\log x}\, dx$$

By Leibnitz's Rule on DUIS, we can diff^n both sides wrt 'α'

$$\frac{d}{d\alpha} I(\alpha) = \frac{d}{d\alpha} \int_0^1 \frac{(x^\alpha - 1)}{\log x}\, dx$$

$$= \int_0^1 \frac{\partial}{\partial \alpha} \frac{(x^\alpha - 1)}{\log x}\, dx$$

$$= \int_0^1 \frac{x^\alpha \cdot \cancel{\log x}}{\cancel{\log x}}\, dx$$

$$= \int_0^1 x^\alpha\, dx$$

Steps:-

① Denote the given integrals by $I(a)$

② Diff both sides wrt 'a'

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③ Integrate RHS wrt 'x'.
④ Integrate now wrt 'α'.
This would give constant of integration C.
⑤ To find C:- Choose a value of 'α' which would make eqn ① zero. Substitute the same value of 'α' in step ④ and find 'C'.

② Type-II:- The ques will be to evaluate an integral and hence deduce some more integral.
① Evaluate the integral using 12th method. ② Using the answer (I) diff. wrt either wrt 'a or b' using DUIS.

Ques① Prove:- $I = \int_0^1 \frac{x^\alpha - x^\beta}{\log x}\,dx = \log\left(\frac{\alpha+1}{\beta+1}\right)$

Sol. Considering 'α' as a parameter.

$$I(\alpha) = \int_0^1 \frac{x^\alpha - x^\beta}{\log x}\,dx \quad \dots ①$$

Using Leibnitz rule on DUIS, we diffⁿ wrt 'α' on both sides:

$$\frac{d}{d\alpha} I(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left[\frac{x^\alpha - x^\beta}{\log x}\right]dx$$

$$= \int_0^1 \frac{x^\alpha \log x}{\log x}\,dx$$

$$= \int_0^1 x^\alpha\,dx$$

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$$

$$\frac{d}{d\alpha} I(\alpha) = \frac{1}{\alpha+1}$$

Integrating wrt 'α'

$$I(\alpha) = \int \frac{1}{\alpha+1}\, d\alpha$$

$$I(\alpha) = \log(\alpha+1) + C \quad \ldots \text{II}$$

To find C:- Put $\alpha = \beta$

$$I(\beta) = \int_0^1 \frac{x^\beta - x^\beta}{\log x} = 0 \qquad \Bigg| \qquad I(\beta) = \log(\beta+1) + C$$

$$C = -\log(\beta+1)$$

Now $I = \log(\alpha+1) - \log(\beta+1)$

$$I = \log\left(\frac{\alpha+1}{\beta+1}\right)$$

Ques ② Prove that: $\displaystyle\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) \sin mx \, dx = \tan^{-1}\left(\frac{b}{m}\right) - \tan^{-1}\left(\frac{a}{m}\right)$

Sol. $\displaystyle I(a) = \int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) \sin mx \, dx \quad \ldots \text{I}$

Using Leibnitz rule on DUIS, we diff wrt "a" on both side;

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$$\frac{d}{da}.I(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\left[\frac{e^{-ax}-e^{-bx}}{x}\right]\sin mx\, dx\right)$$

$$= \int_0^\infty \frac{\sin mx}{x} \cdot e^{-ax}(-x)\, dx$$

$$= -\int_0^\infty e^{-ax}.\sin mx\, dx$$

$$= -\left\{\left[\frac{e^{-ax}}{a^2+m^2}\left[-a\sin mx - m\cos mx\right]\right]\right\}_0^\infty$$

$$= -\left[0 - \left(\frac{1}{a^2+m^2}\right)(-m)\right]$$

$$\frac{d}{da}I(a) = \frac{-m}{a^2+m^2}$$

Integrating wrt 'a'

$$I(a) = -m\int \frac{1}{a^2+m^2}\, da$$

$$I(a) = -m\tan^{-1}\left(\frac{a}{m}\right) + C \quad \ldots II$$

To find C:- Put $a=b$

$$I(b) = \int_0^\infty \left(\frac{e^{-bx}-e^{-bx}}{x}\right)\sin mx\, dx$$

$$= 0$$

From II:-

$$I(b) = -m\tan^{-1}\left(\frac{a}{m}\right) + C$$

$$C = m\tan^{-1}\left(\frac{b}{m}\right)$$

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$$\therefore\ I = \tan^{-1}\left(\frac{b}{m}\right) - \tan^{-1}\left(\frac{a}{m}\right)$$

Ques (3) $$\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx = \cot^{-1}\alpha$$

$$I(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx \quad \cdots I$$

Using Leibnitz rule on DUIS, we diff w.r.t 'α' on both sides,

$$\frac{d}{d\alpha} I(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha} \frac{e^{-\alpha x}\sin x}{x}\,dx$$

$$= \int_0^\infty \frac{e^{-\alpha x}\sin x\,(-x)}{x}\,dx$$

$$= -\int_0^\infty e^{-\alpha x}\sin x\,dx$$

$$= -\left[\frac{e^{-\alpha x}}{\alpha^2+1}\left(-\alpha\sin x - (1)\cos x\right)\right]_0^\infty$$

$$= -\left[0 - \frac{1}{\alpha^2+1}(-1)\right]$$

$$\frac{d}{d\alpha} I(\alpha) = \frac{-1}{\alpha^2+1}$$

$$I\alpha = -\int \frac{1}{\alpha^2+1} = -\tan^{-1}(\alpha) + C \cdots II$$

To find C: Put $\alpha = \infty$

$\{\tan^{-1}x + \cot^{-1}x = \pi/2\}$

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I	II
$I(\infty) = 0$	$I(\infty) = -\tan^{-1}(\infty) + C$
	$C = \pi/2$

$$I(x) = -\tan^{-1}(x) + \frac{\pi}{2}$$

$$= \frac{\pi}{2} - \tan^{-1}(x)$$

$$I(x) = \cot^{-1}(x)$$

Hence Proved

Ques(4) # P.T.:- $\displaystyle\int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}$

Let $\displaystyle I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx$

$$\frac{d}{da} I(a) = \int_0^\infty \frac{1}{x^2} \cdot \frac{1}{(1+ax^2)} \cdot x^2\,dx$$

$$= \int_0^\infty \frac{1}{1+ax^2}$$

$$= \frac{1}{a}\int_0^\infty \frac{1}{x^2+(1/\sqrt{a})^2}\,dx$$

$$= \frac{1}{a}\left[\frac{1}{\frac{1}{\sqrt{a}}}\tan^{-1}\left(\frac{x}{1/\sqrt{a}}\right)\right]_0^\infty$$

$$= \frac{1}{\sqrt{a}}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{1}{\sqrt{a}} \cdot \frac{\pi}{2}$$

$$\therefore\ I(a) = \int \frac{1}{\sqrt{a}} \cdot \frac{\pi}{2}\, da$$

$$= \pi \int \frac{1}{2\sqrt{a}}\, da$$

$$= \frac{\pi}{\cancel{2}} \cdot \frac{a^{1/2}}{1/\cancel{2}}$$

$$I(a) = \pi \cdot \sqrt{a} + C$$

Que ⑤*

$$\int_0^{\pi/2} \frac{\log(1+a\sin^2 x)}{\sin^2 x}\, dx = \pi\left[\sqrt{a+1}-1\right]$$

$$\therefore\ \frac{dI}{da} = \int_0^{\pi/2} \frac{1}{1+a\sin^2 x} \frac{\sin^2 x}{\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + a\tan^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{1+(1+a)\tan^2 x}\, dx$$

$$= \frac{1}{a+1} \int_0^{\pi/2} \frac{\sec^2 x}{\left[\sqrt{1/(1+a)}\right]^2 + \tan^2 x}\, dx$$

Put $t = \tan x$,

$$\frac{dI}{da} = \frac{1}{a+1} \int_0^{\pi/2} \frac{dt}{\left(\sqrt{1/(1+a)}\right)^2 + t^2}$$

$$= \frac{1}{a+1}\left[\sqrt{a+1}\tan^{-1}\left(t\sqrt{a+1}\right)\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{a+1}} \cdot \frac{\pi}{2}$$

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Integrating wrt 'a' :-

$$I = \frac{\pi}{2}\int \frac{da}{\sqrt{a+1}} = \pi\sqrt{a+1} + c$$

Putting $a = 0$,

$$I(0) = \pi + c$$

$$I(0) = \int_0^{\pi/2} \frac{\log 1}{\sin^2 x}\, dx = 0$$

$$[c = -\pi]$$

$$\therefore I = \pi\sqrt{a+1} - \pi$$

$$= \pi\left[\sqrt{a+1} - 1\right]$$

18/03/

Ques 6 # $\int_0^\infty \frac{\cos\lambda x}{x}\left(e^{-ax}-e^{-bx}\right)dx = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right)$

Sol. Let $I(a) = \int_0^\infty \frac{\cos(\lambda x)}{x}\left(e^{-ax}-e^{-bx}\right)dx \quad \ldots \mathrm{I}$

Diff wrt 'a' both sides using L.T.:

$$\frac{dI(a)}{da} = \int_0^\infty \frac{\cos\lambda x}{x}\left(e^{-ax}(-x)\right)dx$$

$$= -\int_0^\infty \cos\lambda x \, e^{-ax}\,dx$$

$$= -\left[\frac{e^{-ax}}{a^2+\lambda^2}\left(-a\cos\lambda x + \lambda\sin\lambda x\right)\right]_0^\infty$$

$$= -\left[0 - \frac{1}{a^2+\lambda^2}(-a)\right]$$

$$\frac{dI(a)}{da} = \frac{-a}{a^2+\lambda^2}$$

$$I(a) = \int \frac{-a}{a^2+\lambda^2}\,da$$

$$= \frac{-1}{2a}\int \frac{2a}{a^2+\lambda^2}\,da$$

$$= \frac{-1}{2a}\log(a^2+\lambda^2) + C \quad \cdots \mathrm{II}$$

To Find C:

Put $a = b$ in

I	II
$I(b) = 0$	$I(b) = \frac{-1}{2b}\log(b^2+\lambda^2) + C$

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$$c = \frac{1}{2}\log(b^2+\lambda^2)$$

$$I(a) = \frac{1}{2}\log(b^2+\lambda^2) - \frac{1}{2}\log(a^2+\lambda^2)$$

$$I(a) = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right)$$

Ques 7 $$\int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x} = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$$

$$I(a) = \int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x}$$

Diff wrt 'a' on both sides using L.T:-

$$\frac{d}{da} I(a) = \left[\frac{\frac{1}{1+x^2/a^2}\cdot\left(\frac{-x}{a^2}\right)}{x}\right]$$

$$= \left\{\frac{\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{-1}{a^2}}{}\right\}$$

$$= \int_0^\infty \left\{\frac{-1}{a^2+x^2}\right\}_0^\infty$$

$$= \left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty$$

$$\frac{d}{da} I(a) = \frac{\pi}{2a}$$

$$I(a) = -\int \frac{\pi}{2a}$$

$$= \frac{-\pi}{2} \int \frac{1}{a}$$

$$I(a) = \frac{-\pi}{2\cancel{a^2}} \log a + C$$

Put $a = b$ in

I

$I(b) = 0$

II

$$I(b) = \frac{-\pi}{2} \log(b) + C$$

$$C = \frac{+\pi}{2} \log(b)$$

$$\therefore I(a) = \frac{\pi}{2} \log\left(\frac{b}{a}\right)$$

Hence proved

Type - II :-

Ques ① Evaluate :-

$$\int_0^{\pi} \frac{dx}{a+b\cos x}$$. Hence evaluate (i) $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2}$

(ii) $\int_0^{\pi} \frac{\cos x \, dx}{(a+b\cos x)^2}$ (iii) $\int_0^{\pi} \frac{dx}{(5+3\cos x)^2}$

Sol. $I = \int_0^{\pi} \frac{dx}{a+b\cos x}$

Put $\tan x/2 = t$

$dx = \frac{2dt}{1+t^2}$

x	0	π
t	0	∞

$\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{a+b\left(\frac{1-t^2}{1+t^2}\right)} = 2\int_0^{\infty} \frac{dt}{a+at^2+b-bt^2}$$

$$= 2\int_0^{\infty} \frac{dt}{(a+b)+(a-b)t^2}$$

$$= \frac{2}{(a-b)} \int_0^{\infty} \frac{dt}{\left(\frac{a+b}{a-b}\right)+t^2}$$

$$= \frac{2}{(a-b)} \int_0^{\infty} \frac{dt}{t^2+\left(\sqrt{\frac{a+b}{a-b}}\right)^2}$$

$$= \frac{2}{(a-b)} \left[\frac{1}{\sqrt{\frac{a+b}{a-b}}} \tan^{-1}\left(\frac{t}{\sqrt{\frac{a+b}{a-b}}}\right) \right]_0^{\infty}$$

$$I = \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$I = \frac{2}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi}{\sqrt{a^2-b^2}}$$

$$\# \left[\frac{\pi}{\sqrt{a^2-b^2}} = \int_0^{\infty} \frac{1}{(a+b\cos x)}\,dx\right] \dots I$$

For deduction:-

(i) Diff (I) w.r.t 'a'

$$\int_0^{\pi} \frac{\partial}{\partial a} \frac{1}{(a+b\cos x)}\,dx = \pi \frac{d}{da}(a^2-b^2)^{-1/2}$$

$$\int_0^{\pi} \frac{-1}{(a+b\cos x)^2}\,dx = \pi\left(\frac{-1}{2}\right)(a^2-b^2)^{-3/2}(2a)$$

$$\int_0^{\pi} \frac{1}{(a+b\cos x)^2}\,dx = \frac{\pi a}{(a^2-b^2)^{3/2}} \dots II$$

(ii) Diff I w.r.t 'b'

$$\int_0^{\infty} \frac{d}{db} \frac{1}{a+b\cos x}\,dx = \pi \frac{d}{db}(a^2-b^2)^{-1/2}$$

$$\int_0^{\infty} \frac{-\cos x}{(a+b\cos x)^2}\,dx = \pi\left(\frac{-1}{2}\right)(a^2-b^2)^{-3/2}(-2b)$$

$$\int_0^{\infty} \frac{\cos x \, dx}{(a+b\cos x)^2} = \frac{-\pi b}{(a^2-b^2)^{3/2}}$$

Put $a=5,\ b=3$ in II

$$I = \frac{5\pi}{(25-9)^{3/2}}$$

$$I = \frac{5\pi}{64}$$

(ii) $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$. Hence deduce.

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2} = \frac{\pi}{ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Numerical Integration

$$\int_a^b f(x)\,dx$$

n = no. of parts / sub-interval.

h = step size / increment in x

$$h = \frac{b-a}{n}$$

x	$y = f(x)$
$x_0 = a$	y_0
$x_1 = a+h$	y_1
$x_2 = a+2h$	y_2
⋮	⋮
$x_n = b$	y_n

Trapezoidal Rule :-

$$\int_a^b y\,dx = \frac{h}{2}\left[\underset{\text{Xtreme}}{X} + 2\underset{\text{Remaining}}{R}\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots y_{n-1})\right]$$

Simpson's $\frac{1}{3}$ rd Rule :-

$$\int_a^b y\,dx = \frac{h}{3}\left[X + 4O + 2E\right]$$

$$= \frac{h}{3}\left\{(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right\}$$

Simpson's $\frac{3}{8}$th Rule:-

$$\int_a^b y\,dx = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6) + 3(y_1+y_2+y_4+\cdots)\right]$$

For T.R.:- n = multiple of one
For S 1/3 Rule :- n = multiple of two
For S 3/8 Rule:- n = multiple of three.
For all methods take $n=6$ (T & C applied)

Ques. $\int_0^1 \frac{x^2}{1+x^3}\,dx$ by dividing into 5 sub intervals using (i) T.R. (ii) S - 1/3rd Rule
Find exact solution

Sol. $a = 0,\ b = 1,\ n = 5$

$h = \frac{b-a}{n} = 0.2$

x	$y = f(x) = x^2/1+x^3$
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.2$	$y_1 = 0.0396$
$x_2 = 0.4$	$y_2 = 0.1503$
$x_3 = 0.6$	$y_3 = 0.296$
$x_4 = 0.8$	$y_4 = 0.4232$
$x_n = 1$	$y_n = 0.5$

* Trapezoidal Rule :-

$$\int_a^b y\,dx = \frac{h}{2}[X + 2R]$$

$$= \frac{h}{2}[(y_0+y_n) + 2(y_1+y_2+y_3+y_4)]$$

$$= \frac{0.2}{2}[\;\cdots\cdots\;]$$

$$\int_0^1 \frac{x^2\,dx}{1+x^2} = 0.2318$$

Simpson's 1/3rd rule

$$\int_a^b y\,dx = \frac{h}{3}[X + 4O + 2E]$$

$$= \frac{h}{3}[(y_0+y_n) + 4(y_1+y_3) + 2(y_2+y_4)]$$

$$\int_0^1 \frac{x^2\,dx}{1+x^3} = 0.19934$$

Exact Solution :-

$$\int_0^1 \frac{x^2\,dx}{1+x^3}$$

$$\frac{1}{3}\int_0^1 \frac{3x^2\,dx}{1+x^3} = \frac{1}{3}[\log(1+x^3)]_0^1$$

$$= \frac{1}{3}[\log 2 - \log 1]$$

$$= \frac{1}{3}(\log_e 2) = 0.23104 \text{ Ans}$$

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Ques ② Compute $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx$ (ln x in Calc)

by (i) T.R. (ii) S 1/3 Rule (iii) S 3/8 Rule
(iv) Compare with exact solution.

Sol. Let $n = 6$

$$h = \frac{b-a}{n} = \frac{1.4-0.2}{6} = 0.2$$

x	$y = \sin x - \log_e x + e^x$
$x_0 = 0.2$	$y_0 = 3.0295$
$x_1 = 0.4$	$y_1 = 2.7975$
$x_2 = 0.6$	$y_2 = 2.8975$
$x_3 = 0.8$	$y_3 = 3.166$
$x_4 = 1.0$	$y_4 = ~~4.0698~~$ 3.5597
$x_5 = 1.2$	$y_5 = 4.0698$
$x_n = 1.4$	$y_n = 4.7041$

(i) TR: $\int_a^b y\,dx = \frac{h}{2}[X + 2R]$

$$= \frac{0.2}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right]$$

$$= 4.07146$$

(ii) S 1/3 Rule:- $\int_a^b y\,dx = \frac{h}{3}[X + 4O + 2E]$

$$= \frac{0.2}{3}\left[(y_0+y_n) + 4(y_3+y_5) + 2(y_2+y_4)\right]$$

$$= 4.05208$$

(iii) S 3/8 Rule :-

$$\int_a^b y\,dx = \frac{3h}{8}\left[X + 2T + 3R\right]$$

$$= \frac{3\times0.2}{8}\left[(y_0+y_n) + 2y_3 + 3(y_1+y_2+y_4+y_5)\right]$$

$$= 4.05293$$

Exact Solution :-

$$\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx$$

$$= \left[-\cos x - x(\log x - 1) + e^x\right]_{0.2}^{1.4}$$

$$= \left[0.810009 + 0.407051 + 4.05519 - 1.221402\right]$$

$$= 4.05084$$

Ques (3)

x	0	1	2	3	4	5	6
$f(x)$	0.146	0.161	0.176	0.190	0.204	0.217	0.230
y	y_0	y_1	y_2	y_3	y_4	y_5	y_n

Evaluate :- $\int_0^6 x \cdot f(x)\,dx$ (i) TR (ii) S 1/3 Rule

$$h = \frac{b-a}{n}$$

$$= \frac{6-0}{6}$$

$$h = 1$$

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x	$f(x)$	$y = x \cdot f(x)$
$x_0 = 0$	0.146	$y_0 = 0$
$x_1 = 1$	0.161	$y_1 = 0.161$
$x_2 = 2$	0.176	$y_2 = 0.352$
$x_3 = 3$	0.190	$y_3 = 0.570$
$x_4 = 4$	0.204	$y_4 = 0.816$
$x_5 = 5$	0.217	$y_5 = 1.085$
$x_n = 6$	0.230	$y_n = 1.38$

(i) **TR:-** $\int_a^b y\,dx = \frac{h}{2}\left[X + 2R\right]$

$$= \frac{1}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= 3.674$$

(ii) **S 1/3 Rule:-**

$$\int_a^b y\,dx = \frac{h}{3}\left[X + 4O + 2E\right]$$

$$\int_0^6 x \cdot f(x)\,dx = \frac{1}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= 3.66$$

Ques 4 Evaluate:- $\int_1^2 e^{-x}\sqrt{x}\,dx$ by taking 7-ordinates.

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Sol. No. of parts (n) = No. of ordinates − 1

$n = 7-1$

$n = 6$

$\therefore h = \frac{b-a}{n} = \frac{1}{6}$

$= \frac{1}{6}$

x	$y = e^{-x}\sqrt{x}$
$x_0 = 0$	$y_0 = 0.3678$
$x_1 = 1/6$	$y_1 = 0.3363$
$x_2 = 2/6$	$y_2 = 0.3043$
$x_3 = 3/6$	$y_3 = 0.2732$
$x_4 = 4/6$	$y_4 = 0.2438$
$x_5 = 5/6$	$y_5 = 0.2164$
$x_n = 1$	$y_n = 0.1913$

(i) TR:- $\int_a^b y\,dx = \frac{h}{2}\left[x+2R\right]$

$= \frac{1}{12}\left[(y_0+y_n)+2\left[y_1+y_2+y_3+y_5+y_4\right]\right]$

$= 0.2755$

(ii) S 1/3 Rule:- $\int_a^b y\,dx = \frac{h}{3}\left[X+4O+2E\right]$

$= \frac{1}{18}\left[(y_0+y_n)+4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$

$= 0.2754$

Ques ⑤ Evaluate :- $\int_0^1 \frac{1}{1+x^2}\,dx$ by S1/3 and S3/8 Rule

Hence obtain the approximate value of y in each case.

Sol. $h = \frac{b-a}{n} = \frac{1-0}{6}$

$= \frac{1}{6}$

x	$y = 1/1+x^2$
$x_0 = 0$	$y_0 = 1$
$x_1 = 1/6$	$y_1 = 0.9729$
$x_2 = 2/6$	$y_2 = 0.9$
$x_3 = 3/6$	$y_3 = 0.8$
$x_4 = 4/6$	$y_4 = 0.6923$
$x_5 = 5/6$	$y_5 = 0.5901$
$x_n = 1$	$y_n = 0.5$

S-1/3 Rule :-

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[X + 2E + 4O\right]$$

$$= \frac{1}{18}\left[(y_0+y_n) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.7853 \quad \dots ①$$

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S - 3/8 Rule :-

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{3h}{8}\left[X + 2T + 3R \right]$$

$$= \frac{3\times 1}{8\times 6}\left[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5) \right]$$

$$= 0.7853 \quad \cdots (ii)$$

Exact Solution :-

$$\int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x \right]_0^1$$

$$= \frac{\pi}{4} \quad \cdots (iii)$$

From eqn (i) and (iii)

$$0.7853 = \frac{\pi}{4}$$

$$\pi = 3.1412$$

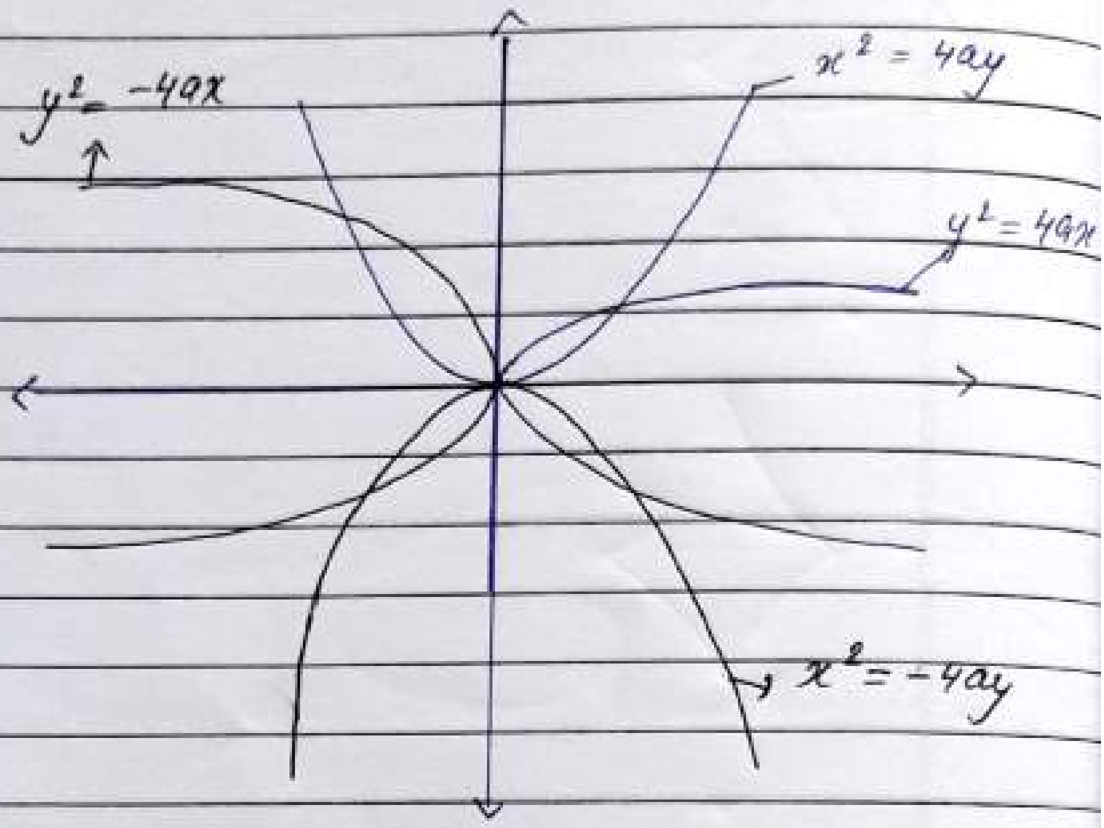
From eqn (ii) and (iii)

$$0.7853 = \frac{\pi}{4}$$

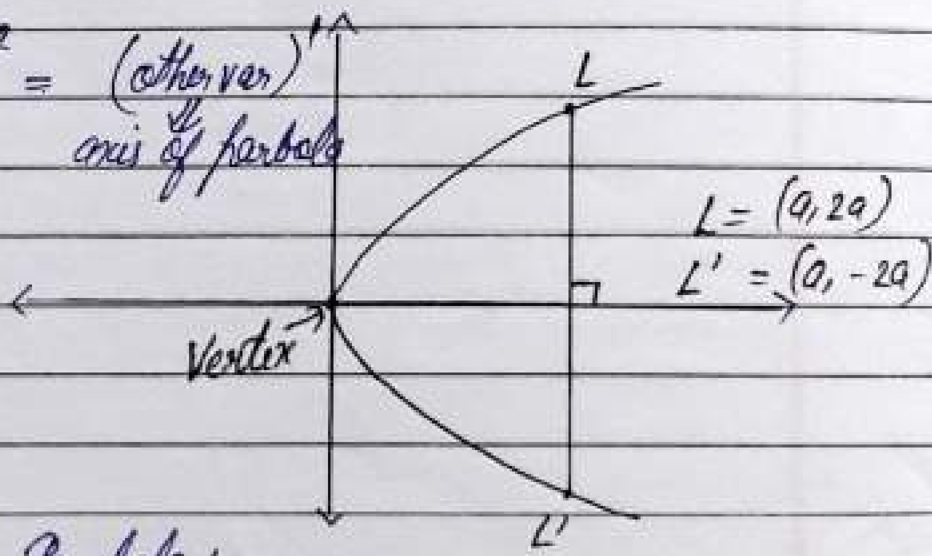
$$\pi = 3.1412$$

Summary of Curves:-

① **Parabola:-** $y^2 = -4ax$, $x^2 = 4ay$, $x^2 = -4ay$
$y^2 = 4ax$



$(\text{One Ver})^2 = \underbrace{(\text{other ver})^1}_{\text{axis of parabola}}$



Shifted Parabola:-

$$(x-h)^2 = 4a(y-k)$$

Vertex (h, k)

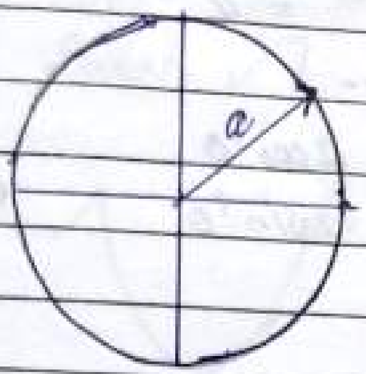
Example:- $y = x^2 - x + \frac{1}{4} - \frac{1}{4}$

$$\left(y - \frac{1}{4}\right) = \left(x - \frac{1}{2}\right)^2$$

② Circles:-

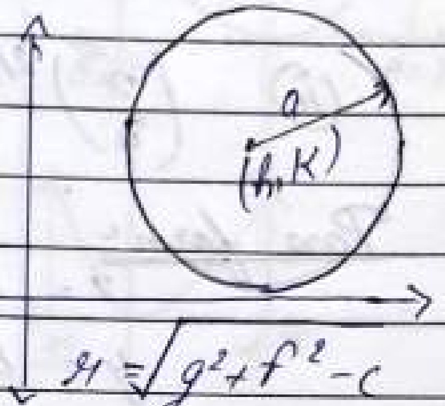
① Standard Circle:-

$x^2+y^2=a$



(ii) General Circle:-

(a) $(x-h)^2+(y-k)^2=a^2$



(b) $x^2+y^2+2gx+2fy+c=0$

$c=(-g,-f)$

$r=\sqrt{g^2+f^2-c}$

③ Loops:-

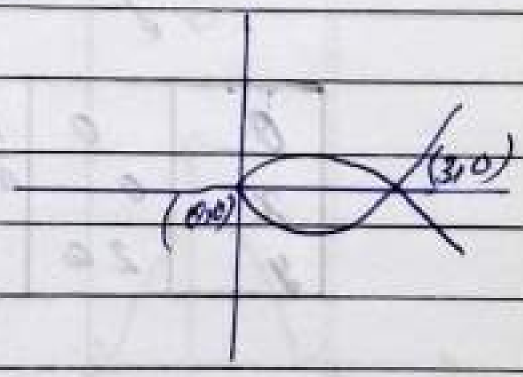
A loops is formed when by putting $y=0$. We ~~d~~ get two values of x and the curve is not defined beyond any one pt.

① $y^2=x(x-3)^2$

(i) Put $y=0,\ x=0,\ x=3.$

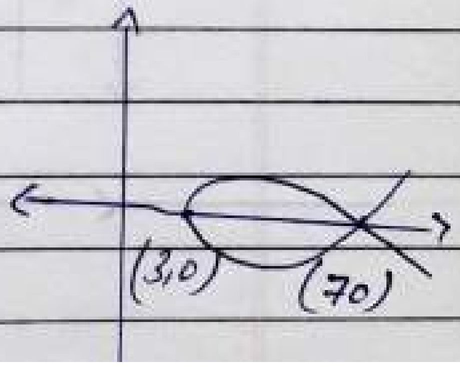
(ii) $x>3,\ y^2=+ve$ ✓

$x<0,\ y^2=-ve$ ✗



② $y^2=(x-3)(x-7)^2.$

$y=0,\ x=3\ \&\ 7.$



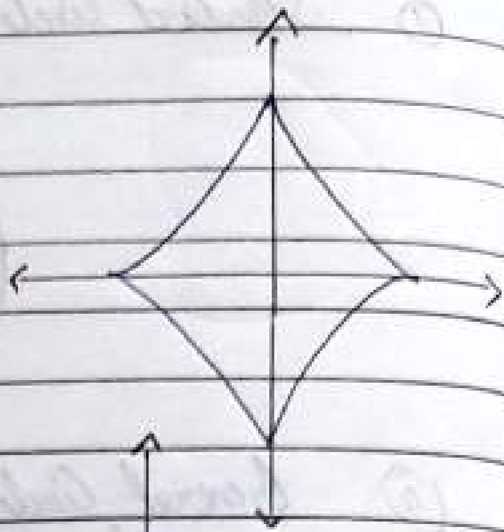
④ Astroid:-

(i)

$$x^{2/3}+y^{2/3}=a^{2/3}$$

Para form:-

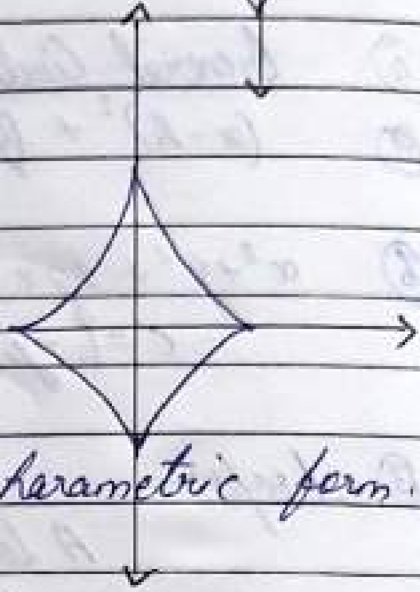
$$x = a\cos^3\theta$$
$$y = a\sin^3\theta$$



(ii) $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

Para form:-

$$x = a\cos^3\theta$$
$$y = b\sin^3\theta$$

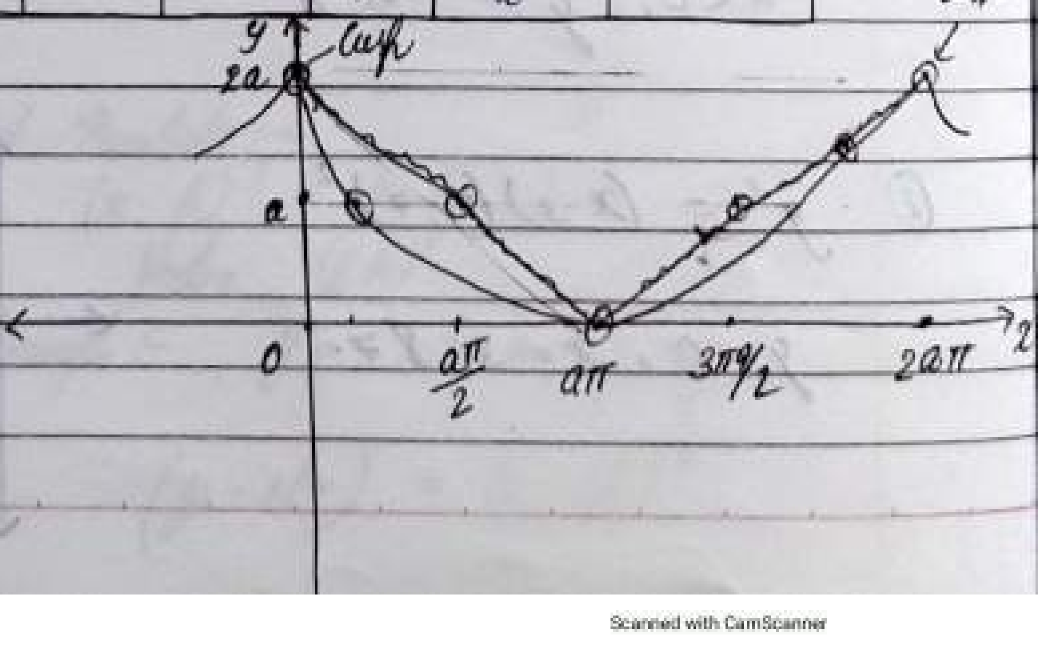


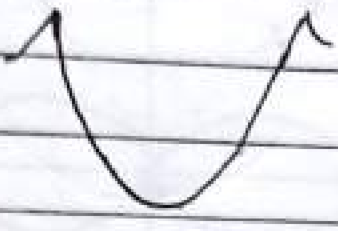
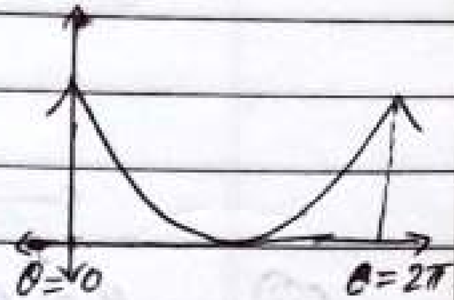
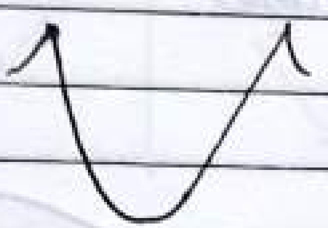
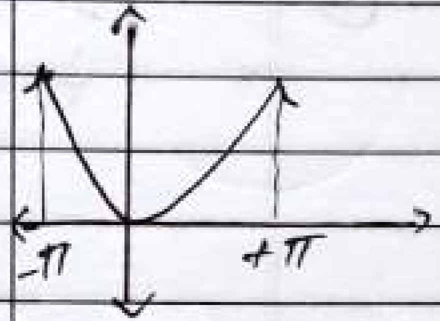
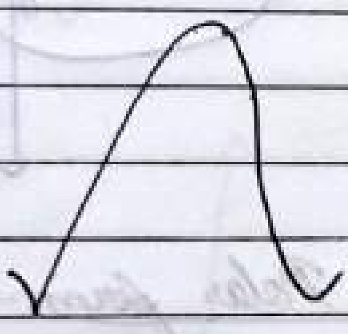
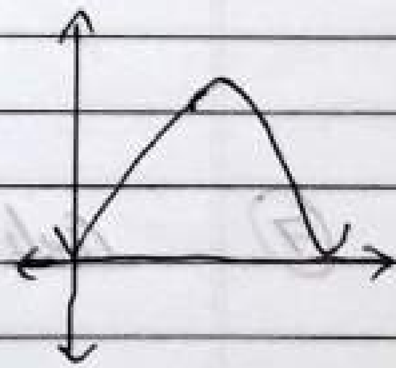
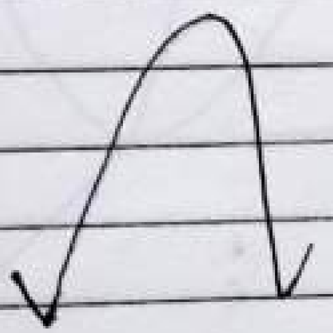
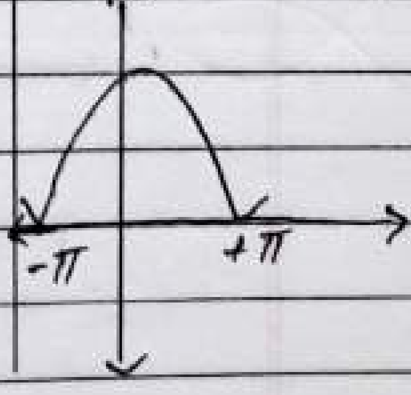
Always solve Astroid parametric form.

⑤ Cycloid:-

$$x = a(\theta + \sin\theta)$$
$$y = a(1+\cos\theta)$$

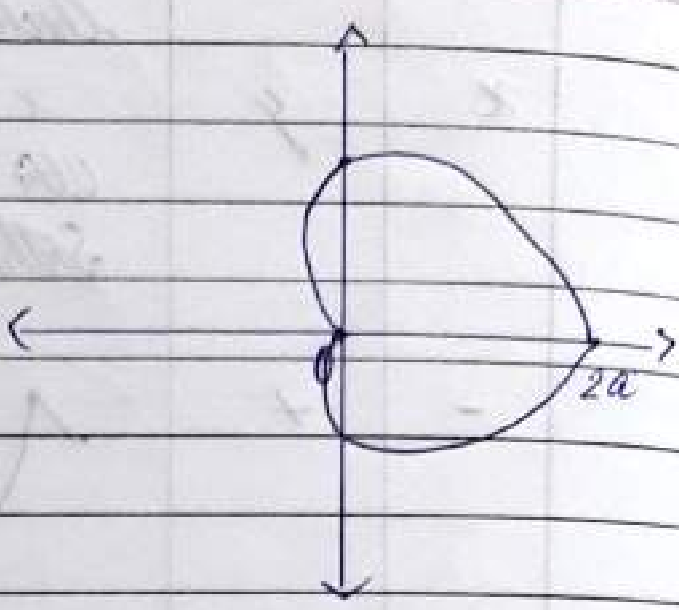
θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2-1)$	$a\pi$	$a(3\pi/2+1)$	$a\times2\pi$
y	$2a$	a	0	a	$2a$



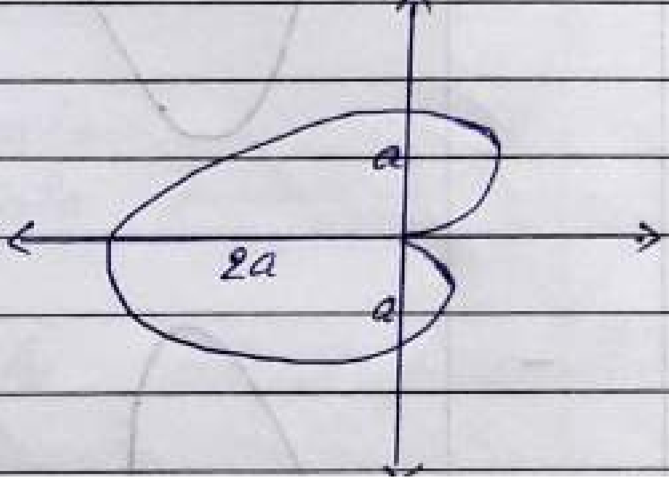
x	y	Check sign of x & y Ulta:- Sidha:-	Check the sign of x -ve:- Entirely in +ve +ve:- half +ve half -ve.
-	+		
+	-		
-	-		
+	+		

(6) Cardioide:-

(i) $r = a(1+\cos\theta)$



(ii) $r = a(1-\cos\theta)$



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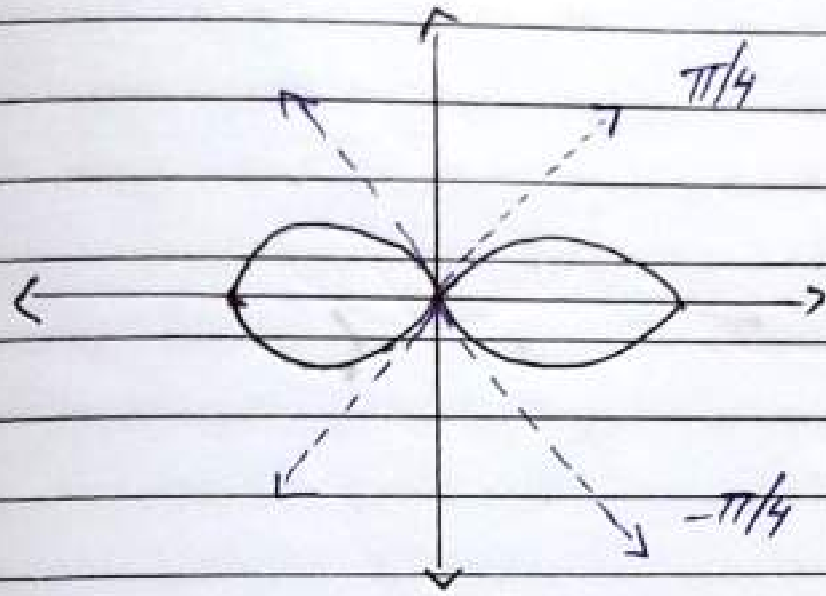
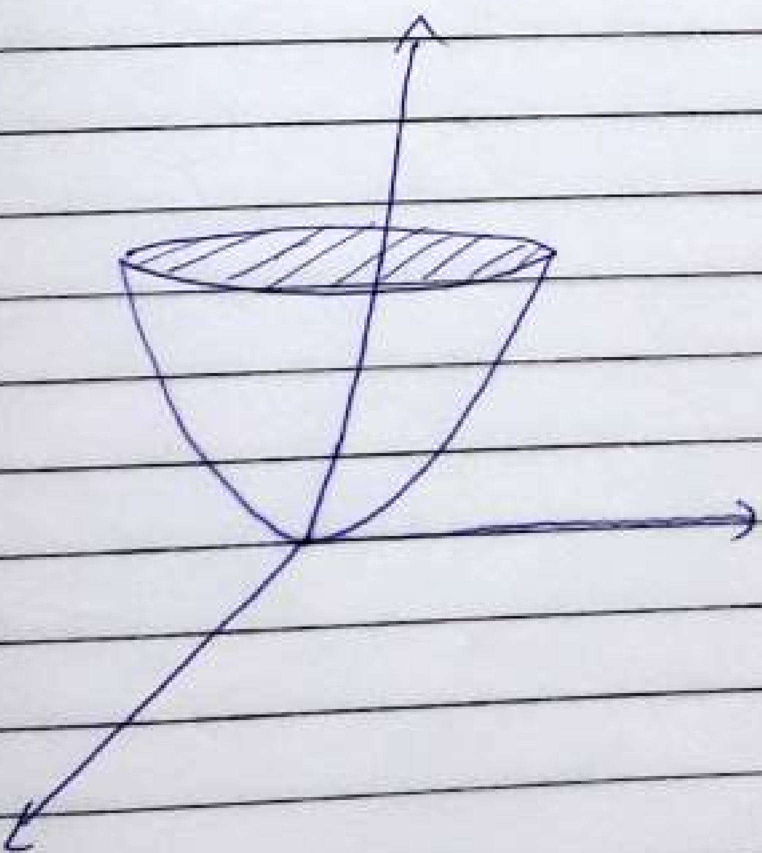
(8) Bernoulli's Lemniscate :-

$$r^2 = a^2 \cos 2\theta$$



Paraboloide:-

$$x^2 + y^2 = az$$



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Rectification

Rectification means to find arc length b/w two pts on curve.

To find arc length, the following will be given :-

(i) Eqn of curve will be given in cartesian/ parametric / polar form.

(ii) Two pts on the curve will be given directly or indirectly.

(iii) Formulas to find arc length:

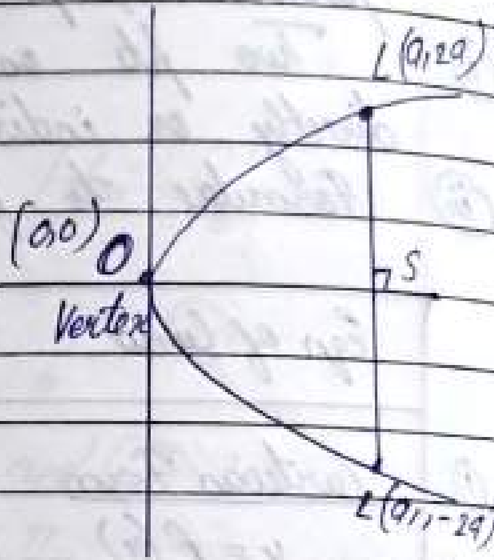
Eqn of Curve	Arc length
(1) Cartesian Form $y = f(x)$ or $x = f(y)$	$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ Or $S = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$
(2) Parametric Form :- $x = f(\theta)$ $y = g(\theta)$	$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$
(3) Polar Form $r = f(\theta)$	$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Exercise I (Cartesian Form)

Ques 1 Find the arc length of parabola $y^2 = 4ax$ from the vertex to the end of lactus rectum. Also find the arc length cut-off by the line $3y = 8x$.

Sol. Part - I

$$y^2 = 4ax$$
$$x = \frac{y^2}{4a}$$



To Find :- $l(OL)$

Sol. $\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

Required arc length :-

$$S = l(\text{arc } OL)$$
$$= \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^{2a} \sqrt{1 + \left(\frac{y}{2a}\right)^2}\, dy$$

$$= \frac{1}{2a} \int_0^{2a} \sqrt{y^2 + (2a)^2}\, dy$$

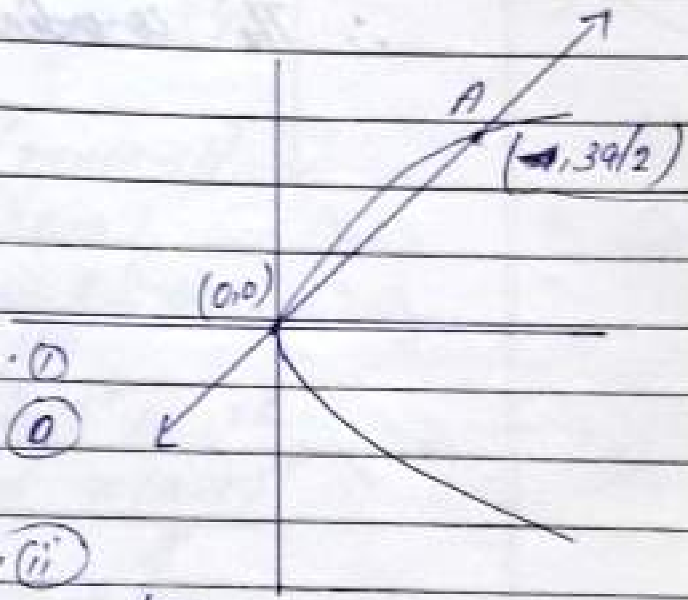
$$= \frac{1}{2a} \left[\sqrt{y^2 + 4a^2} + \frac{(2a)^2}{2} \log \left(y + \sqrt{y^2 + 4a^2}\right) \right]_0^{2a}$$

$$S = \frac{1}{2a}\left[\sqrt{(2a)^2+4a^2} + 2a^2\log\left(2a+\sqrt{4a^2+4a^2}\right) - \left\{\sqrt{0+4a^2} + 2a^2\log\left(0+\sqrt{0+4a^2}\right)\right\}\right]$$

$$= \frac{1}{2a}\left[2a\sqrt{2} + 2a^2\log(2a+2a\sqrt{2}) - \left(2a + 2a^2\log(2a)\right)\right]$$

$$S = a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Part II :-



To find A :-

$3y = 8x \quad \ldots (1)$

$y = \frac{8x}{3} \quad \ldots (i)$

$y^2 = 4ax \quad \ldots (ii)$

From eqn (i) and (ii)

$$\frac{64x^2}{9} = 4ax$$

$$64x^2 = 36ax$$

$$\therefore x(64x - 36a) = 0$$

$$x = 0 \quad \text{or} \quad 64x - 36a = 0$$

$$\therefore 64x - 36a = 0$$

$$x = \frac{36a}{64} = \frac{9a}{16}$$

From eqn (1)

$$3y = 8x$$

Putting $x = 0$; $y = 0$.

Putting $x = \frac{9a}{16}$

$$3y = \frac{8 \times 9a}{16}$$

$$y = \frac{9a}{2 \times 3}$$

$$y = \frac{3a}{2}$$

$\therefore$ The co-ordinates of A are $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ Ans

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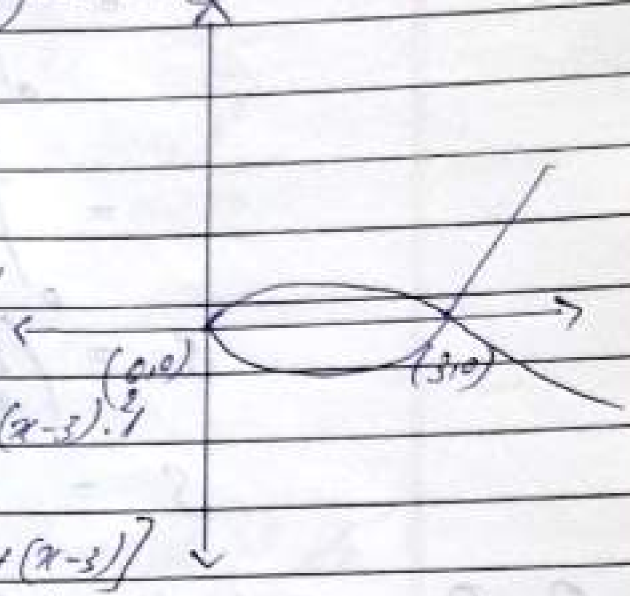
Ques ② Find the loop (length of) $9y^2 = x(x-3)^2$.

Sol. $9y^2 = x(x-3)^2 \cdots ①$

$y = 0, \; x = 0, 3.$



Diff eqn ① wrt 'x'

$18y\dfrac{dy}{dx} = x \cdot 2(x-3) + (x-3)^2 \cdot 1$

$= (x-3)\left[2x + (x-3)\right]$

$= (x-3)(3x-3)$

$\dfrac{dy}{dx} = \dfrac{(x-3)\,3(x-1)}{18y}$

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{\cancel{(x-3)^2}(x-1)^2}{4 \cdot 36 \dfrac{x\cancel{(x-3)^2}}{9}}$

$= \dfrac{(x-1)^2}{4x}$

Now

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{(x-1)^2}{4x}$

$= \dfrac{(x+1)^2}{4x}$

$S = l(\text{arc } ABCDA)$

$= 2\,l(\text{arc } ABC)$

$= 2\int_{x_1}^{x_2} \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$

$= 2\int_0^3 \sqrt{\dfrac{(x+1)^2}{4x}}\, dx$

$$S = \int_0^3 \frac{x+1}{\sqrt{x}}\,dx$$

$$= \int_0^3 \frac{x}{\sqrt{x}}\,dx + \int_0^3 \frac{1}{\sqrt{x}}\,dx$$

$$= \int_0^3 x^{1/2}\,dx + \int_0^3 x^{-1/2}\,dx$$

$$S = \left(\frac{2}{3}(3)^{1/2}\cdot 3 + 2(3)^{1/2}\right)$$

$$S = 4\sqrt{3}$$

Ques ⑧. Find the total perimeter of the loop $9y^2 = (x+7)(x+4)^2 \cdots ①$

Sol. Put $y = 0$, $x = -7, -4$.

Diff eqn ① wrt x,

$$18y\frac{dy}{dx} = (x+7)\cdot 2(x+4) + (x+4)^2\cdot 1$$

$$18y\frac{dy}{dx} = x+4\left((x+7)2 + (x+4)\right)$$

$$= (x+4)\left[2x+14+x+4\right]$$

$$= (x+4)\left[3x+18\right]$$

$$\frac{dy}{dx} = \frac{[x+4]^2\cdot 3(x+6)^2}{18_6 y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\cancel{(x+4)}^2(x+6)^2}{36\frac{(x+7)(x+4)^2}{9}}$$

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$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+6)^2}{4(x+7)(x+7)}$$

$$= \frac{(x+6)^2}{4x+28}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x+6)^2}{4x+28}$$

$$= \frac{(x+8)^2}{4(x+7)}$$

$$S = 2\int_{-7}^{-4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_{-7}^{-4}\left(\sqrt{x+7} + (x+7)^{-1/2}\right)dx$$

$$= \left[\frac{2}{3}(x+7)^{3/2} + 2(x+7)^{1/2}\right]_{-7}^{-4}$$

$$S = 4\sqrt{3} \quad \underline{\underline{Ans}}$$

Ques(2) Find the length of the arc of the curve $y = \log(e^x-1) - \log(e^x+1)$ from $x=1$ to $x=2$.

{ Note:- logarithmic curves are in syllabus ? Hence in exam diagram is not expected. }

Sol.

$$\frac{dy}{dx} = \frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}$$

$$= \frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x-1)(e^x+1)}$$

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$$\frac{dy}{dx} = \frac{e^{3x}+e^{x}-e^{2x}+e^{-x}}{e^{2x}-1}$$

$$\frac{dy}{dx} = \frac{e^{x}}{e^{2x}-1}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{e^{2x}}{(e^{2x}-1)^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{e^{4x}-2e^{2x}-1+e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{2x}}{e^{2x}-1} - \frac{1}{(e^{2x}-1)^2}$$

$$S = l(\text{arc})$$

$$= \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \left(\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right)$$

$$= \log\left(\frac{e^{x}-e^{-x}}{2}\right)_1^2 = \log\left(\frac{e^4-1}{(e^2-1)}\right)$$

$$S = \log\left(e+\frac{1}{e}\right) \quad \underline{\underline{\text{Ans}}}$$

<u>H/W</u>

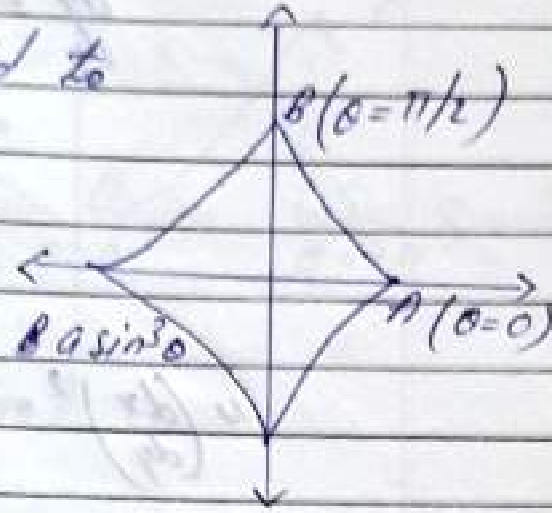
Ques ⑤ Find the length of the arc of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y=1$ to $y=2$

28/03/19

* Parametric Form

Board: Show that perimeter of Astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $6a$.

Sol. Converting Astroid to parametric form.



$\therefore \; x = a\cos^3\theta$

$y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2$$

$$= 9a^2\cos^4\theta\sin^2\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = 9a^2\sin^4\theta\cos^2\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$(\cos^2\theta + \sin^2\theta = 1)$

$$= 9a^2\sin^2\theta\cos^2\theta$$

$$S = 4\int_0^{\pi/2} \sqrt{9a^2\cos^2\theta\sin^2\theta}\, d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

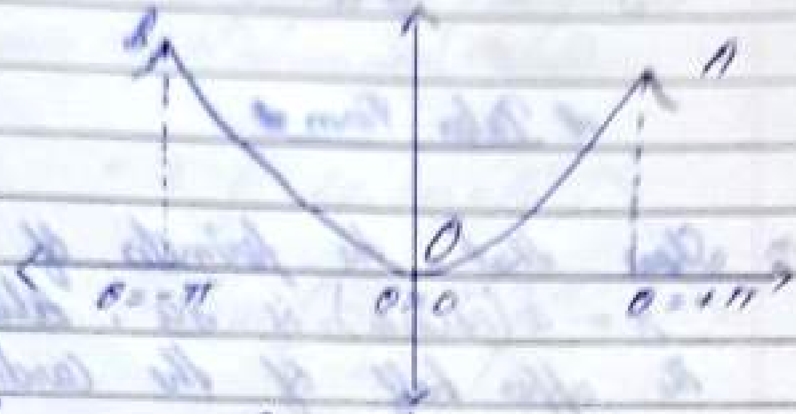
$$= 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$S = 6a$ Ans

29/08/19

* Parametric Form \#

Ques (2) Show that the length of cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ from cusp to cusp is $8a$.



$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$

$$= a^2\sin^2\theta + a^2 + 2a^2\cos\theta + \cos^2\theta\, a^2$$

$$= a^2(1 + 2\cos\theta)$$

$$= 4a^2\cos^2\theta/2$$

$$S = 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= 2\int_0^{\pi} \sqrt{4a^2\cos^2\theta/2}$$

$$= 2\int_0^{\pi} 2a\cos\theta/2$$

$$= 4a\int_0^{\pi} \cos\theta/2$$

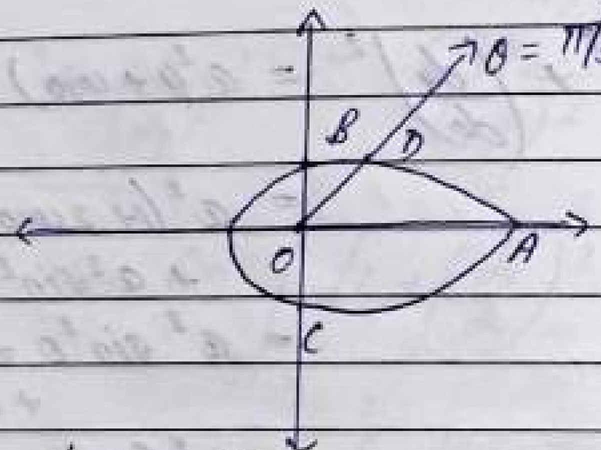
$$S = 4a\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi}$$

$$S = 8a$$

Polar Form

Ques① Show that the perimeter of cardioide $r = a(1+\cos\theta)$ is $8a$. Also show that the upper half of the cardioide is bisected at $\theta = \pi/3$.

Sol. $r = a(1+\cos\theta)$



$$\frac{dr}{d\theta} = -a\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta$$

$$= a^2 + 2a^2\cos\theta + a^2(\sin^2\theta+\cos^2\theta)$$

$$= 2a^2(1+\cos\theta)$$

$$= 4a^2\cos^2\theta/2$$

Part I:-

Required arc length:-

$$S = l(\text{arc } ABOCA)$$

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$$S = 2\, l(ABO)$$

$$= 2 \cdot \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2 \int_{0}^{\pi} 2a \cos\theta/2$$

$$= 4a \left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi}$$

$$= 8a$$

Part-II :-

$$S = l(\text{arc } AD)$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2a \int_{0}^{\pi/3} \left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi/3}$$

$$= 4a \left[\sin\frac{\pi}{3/2} - \sin 0\right]$$

$$= 4a \left[\frac{1}{2} - 0\right]$$

$$= 2a$$

$$S = l(\text{arc } DO)$$

$$= 2a \int_{\pi/3}^{\pi} (\cos\theta/2)$$

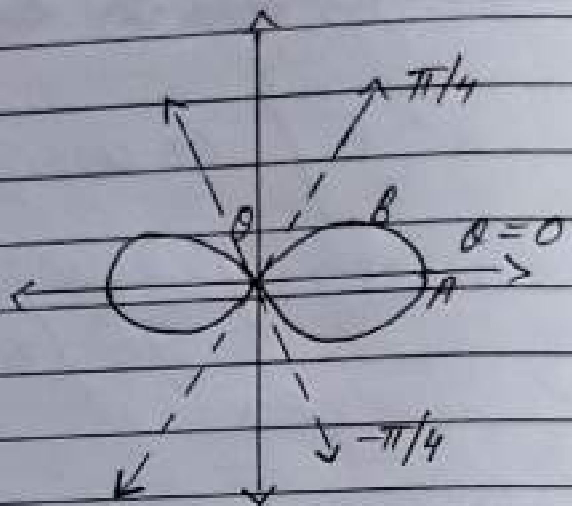
$$= 2a \left[\frac{\sin\theta/2}{1/2}\right]_{\pi/3}^{\pi}$$

$$= 2a.$$

Date 01/04/19

Ques Show that the perimeter of Bernoulli's lemniscate $r^2 = a^2\cos 2\theta$ is $\frac{a}{\sqrt{2\pi}}\left(\Gamma\frac{1}{4}\right)^2$

Sol.



$$r^2 = a^2\cos 2\theta$$

$$r = a\sqrt{\cos 2\theta}$$

$$\frac{dr}{d\theta} = \frac{a}{2\sqrt{\cos 2\theta}} \cdot (-\sin\theta 2)\cdot 2$$

$$= \frac{-a\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2(\sin 2\theta)^2}{\cos 2\theta}$$

$$= \frac{a^2\cos 4\theta + a^2\sin^2 2\theta}{\cos 2\theta}$$

$$= \frac{a^2}{\cos 2\theta}$$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Required arc length:

$$S = 4\, l(\text{arc } ABO)$$

$$= 4\int_0^{\pi/4} \sqrt{\frac{a^2}{\cos 2\theta}}\, d\theta$$

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$$= 4\int_0^{\pi/4} \frac{a}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 4a \int_0^{\pi/4} \cos^{-1/2} 2\theta \, d\theta$$

Put $2\theta = t$

θ	0	$\pi/4$
t	0	$\pi/2$

$$d\theta = dt/2$$

$$S = 4a \int_0^{\pi/2} \cos^{-1/2} t \, \frac{dt}{2}$$

$$= \frac{4a}{2} \cdot \left\{ \frac{1}{2}\, \beta\left(\frac{p+1}{2}, \frac{-\frac{1}{2}+1}{2} \right) \right\}$$

$$= a\, \beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= a\, \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma 3/4}$$

$$= \frac{a\sqrt{\pi}\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}$$

$$= \frac{a\sqrt{\pi}\,(\Gamma 1/4)^2}{\sqrt{2}\,\pi}$$

$$S = \frac{a\left(\Gamma\frac{1}{4}\right)^2}{\sqrt{2\pi}} \quad \underline{\underline{Ans}}$$

Multiple Integration

Date 01/04/19

Double Integration in Cartesian Form:-

(1) When the limits of integration are given

$$I = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$$

To evaluate the above integral check the limits of inner most integral

(i) Here, the inner most integral has limit as a fn of x. Hence, they are y limits i.e from $y = f_1(x)$ to $y = f_2(x)$

(ii) Once this is done we evaluate using the outer limit from $x = a$ to $x = b$.

(2) When the limits of integration are not given:

(i) Here, we take an elementary strip ∥ to x-axis or y-axis. Consider y-axis. The curves where the lower end & upper end of the strip touches gives y limits

Therefore, y limits: $y = f_1(x)$ to $y = f_2(x)$

(ii) Now move the strip in x-direction. Therefore, the corner pts of the region give x-limits.

i.e. $x = a$ to $x = b$

Right their limits from inside to outside.

$$I = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

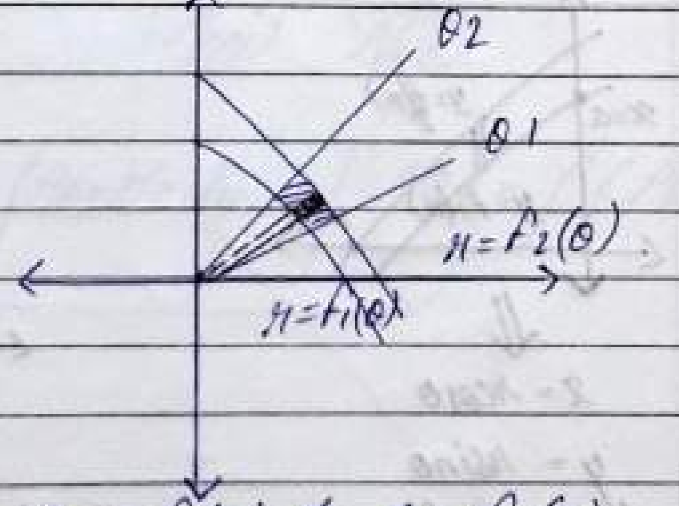
$$\boxed{\text{Strip} \parallel \text{to } y\text{-axis} \longrightarrow \text{Write } y \text{ limits first} \Rightarrow \int \text{ first w.r.t } y}$$

\# The criteria to strip, ||el to x-axis / y-axis is as follows:-

Priority 1:- If the given integral is easy to solve with x first, then strip ||el to x-axis and vice-versa.

Priority 2:- Take strip ||el in such a way we get minimum no. of regions.

II Double Integration Using Polar Form:



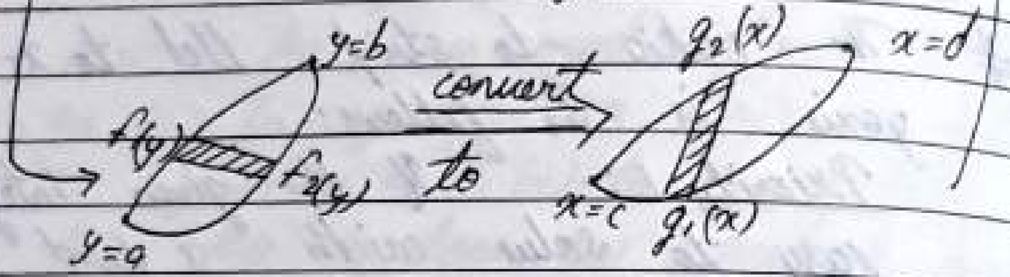
r-limits: $r = f_1(\theta)$ to $r = f_2(\theta)$

θ-limit: $\theta = \theta_1$ to $\theta = \theta_2$ in anticlockwise.

$$\int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} dr\,d\theta$$

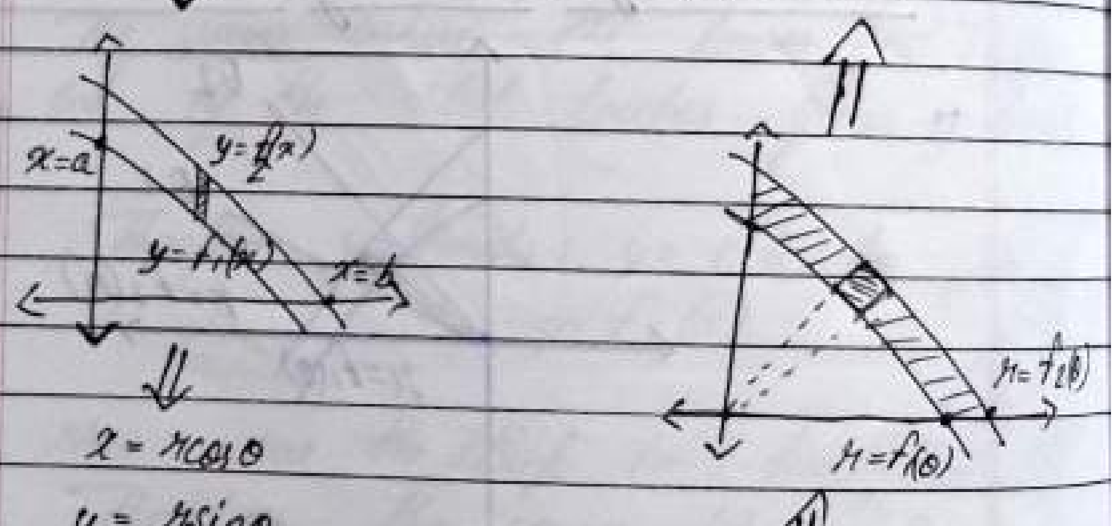
Changing the Order of Integration :-

$$\int_a^b\int_{f_1(y)}^{f_2(y)} F(x,y)\,dx\,dy = \int_c^d\int_{g_1(x)}^{g_2(x)} F(x,y)\,dy\,dx$$



Change Cartesian to Polar Form :-

$$\int_a^b\int_a^{f_2(x)} F(x,y)\,dy\,dx = \int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$ $\Longrightarrow$ Find eqn of curves in polar form

$dx\,dy = r\,dr\,d\theta$

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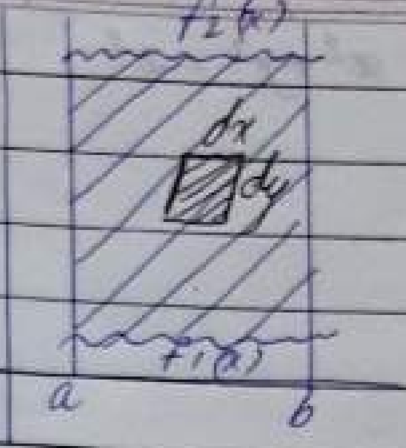


Cartesian Form:

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$$

Polar Form:-

$$A = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta.$$

Mass of Thin Area:

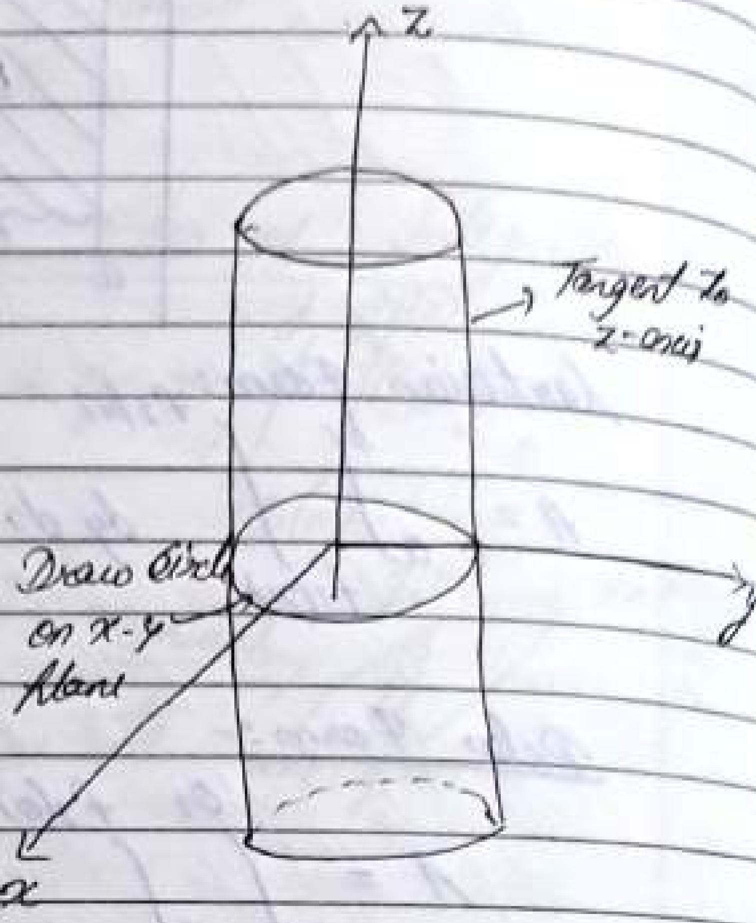
Mass = Density × Area

$\rho(x,y)$ $\rho(r,\theta)$

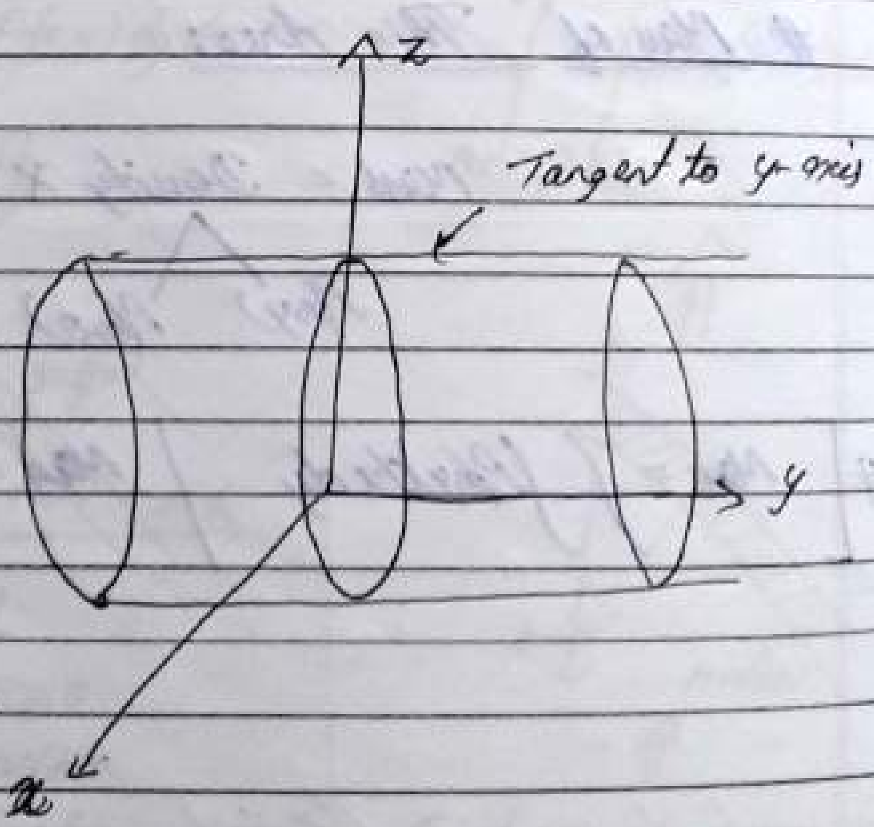
[Mass $= \iint \rho(x,y)\, dx\, dy$ | Mass $= \iint \rho(r,\theta)\, r\, dr\, d\theta$]

Cylinder Volume##

(1) $x^2+y^2=a^2$



(2) $x^2+z^2=a^2$



Double Integration in Cartesian Form

(1) When limits of integration are given:-

Ques (1)

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2-x^2}}\, dx\, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{(\sqrt{1-y^2})^2 - x^2}\, dx\, dy$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \sin^{-1}(1) - \sin^{-1}(0)]\, dy$$

$$= \frac{\pi}{2} \int_0^1 1\, dy$$

$$= \frac{\pi}{2} [y]_0^1$$

$$I = \pi/2$$

Ques (2)

$$I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x\, dy}{y^2+x^2+a^2}\, dx$$

$$= \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} x \frac{1}{(\sqrt{x^2+a^2})^2+y^2}\, dy\, dx$$

$$= \int_0^{a\sqrt{3}} x \left[\frac{1}{\sqrt{x^2+a^2}} \tan^{-1}\left(\frac{y^2}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}}$$

$$= \frac{\pi}{4} \int_0^{a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\, dx$$

$$I = \frac{\pi a}{4}$$

Ques (3) $$I = \int_0^1 \int_0^y y x e^{-x^2} dx\, dy$$

$$= \int_0^1 y \int_0^y e^{-x^2} x \, dx\, dy$$

$$= \frac{-1}{2} \int_0^1 y \int_0^y e^{-x^2}(-2x)\, dx\, dy$$

$$= \frac{-1}{2} \int_0^1 y \left[e^{-x^2}\right]_0^y dy$$

$$= \frac{-1}{2} \int_0^1 e^{-y^2} y\, dy$$

$$= \frac{-1}{2} \int_0^1 y(e^{-y^2} - 1)\, dy$$

$$= \frac{-1}{2} \int_0^1 y e^{-y^2} - y\, dy$$

$$= \frac{-1}{2} \cdot \frac{-1}{2} \left[\int_0^1 e^{-y^2}(-2y)\, dy - \int_0^1 y\, dy\right]$$

$$= \frac{1}{4} \left\{ \left(e^{-y^2}\right)_0^1 - \left[\frac{y^2}{2}\right]_0^1 \right\}$$

$$= \frac{1}{4} \left[e^{-1} - 1 + \frac{1}{2}\right]$$

$$I = \frac{1}{4e} \text{ Ans}$$

Ques (4) $I = \int_0^{\infty} dx \int_0^1 e^{-x^a y} dy$

$$= \int_{x=0}^{\infty} \int_{y=0}^{1} e^{-x^a y} dy\, dx$$

<u>Note:-</u> All the limits of integration are zero. Hence we can solve the integration in any order.

$$I = \int_0^{\infty} \left[\frac{e^{-x^a y}}{-x^a} \right]_0^1 dx$$

$$= - \int_0^{\infty} \left[x^{-a} \left(e^{-x^a} - 1 \right) \right] dx$$

$$= \int_0^{\infty} \left[x^{-a} - e^{-x^a} x^{-a} \right] dx$$

$$= \left[\frac{x^{-a+1}}{-a+1} \right]_0^{\infty} - \int_0^{\infty} e^{-x^a} x^{-a} dx$$

(In type-I)

Put $x^a = t$

$x = t^{1/a}$

$dx = \frac{1}{a} t^{1/a - 1} dt$

$$I = I_2 = -\int_0^\infty e^{-t}\, t^{-1}\, \frac{t^{\frac{1}{a}-1}}{a}\, dt$$

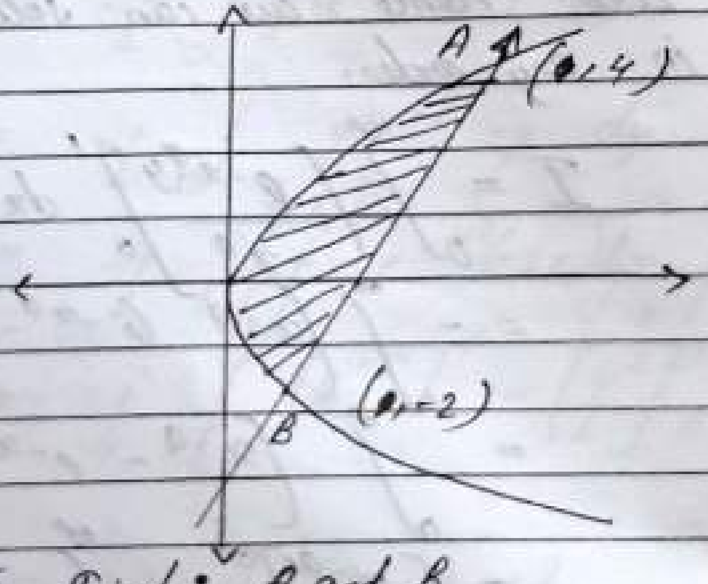
$$I_2 = \frac{-1}{a}\int_0^\infty e^{-t}\, t^{\left(\frac{1}{a}-1\right)-1}\, dt$$

$$I_2 = \frac{-1}{a}\,\Gamma\!\left(\frac{1}{a}-1\right)$$

II # When the limits of Integration are not given :-

Ques ① Evaluate :

$\iint xy\,dx\,dy$ over the area of the triangle bounded by $y^2 = 4x$ & $y = 2x - 4$ $<\begin{matrix}(2,0)\\(0,-4)\end{matrix}$



To Find : A and B

$$y = 2x - 4 \quad \ldots①$$
$$y^2 = 4x$$
$$(2x-4)^2 = 4x$$
$$4x^2 - 16x - 16 = 4x$$
$$4x^2 - 20x - 16 = 0$$
$$x^2 - 5x - 4 = 0$$
$$x\ (4, 1)$$

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$y = -2$ (Putting $x=1$ in eqn (1))
$y = +4$ (Putting $x=4$ in eqn (1))

(Curves) x- limits:-

$$x = \frac{y^2}{4} \text{ to } x = \frac{y+4}{2}$$

(Corner pts) y- limits:- $y = -2$ to 4

$$I = \int_{-2}^{4}\int_{y^2/4}^{y+4/2} xy\,dx\,dy$$

$$= \int_{-2}^{4}\left[\int_{y^2/4}^{y+4/2} x\,dx\right] y\,dy$$

$$= \int_{-2}^{-4}\left[\frac{x^2}{2}\right]_{y^2/4}^{y+4/2} y\,dy$$

$$= \int_{-2}^{4}\left(\frac{(y+4)^2}{4\cdot 2} - \frac{y^4}{16\cdot 2}\right) y\,dy$$

$$= \frac{1}{8}\int_{-2}^{4} y\,(y+4)^2 - \frac{y^5}{4}\,dy$$

$$= \frac{1}{8}\int_{-2}^{4} y\,(y^2+8y+16) - \frac{y^5}{4}\,dy$$

$$= \frac{1}{8}\int_{-2}^{4}\left[\frac{y^4}{4} + \frac{8y^3}{3} + \frac{16y^2}{2} - \frac{y^6}{6}\right]_{-2}^{4}$$

$$= \frac{45}{2} \text{ Ans}$$

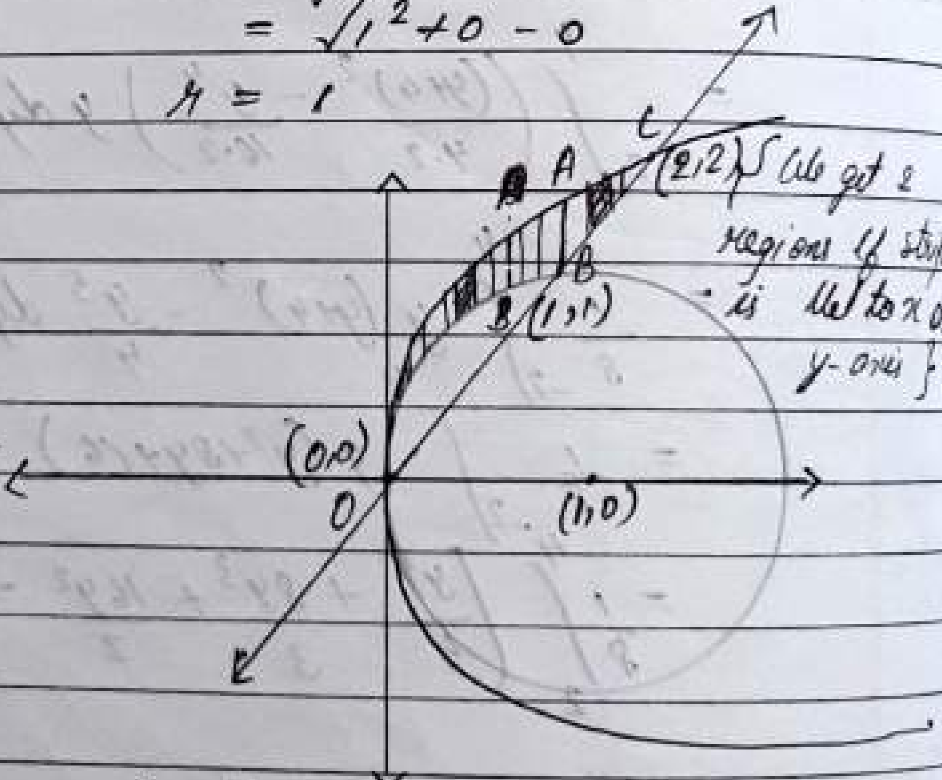
Date 02/04/19

Take an elementary strip ∥ to x-axis. The curves where the left end and the right end touches gives ~~g~~ x-limits. Then the move the strip in y-direction (corner to corner)

Ques ② Evaluate: $\iint_R xy\,dx\,dy$ where R is the region bounded by
(i) $x^2+y^2-2x=0$
(ii) $y^2=2x$
(iii) $y=x$

$y=\sqrt{2x-x^2}$
$=\sqrt{x(2-x^2)}$
$y=\sqrt{2x}$

Sol. $x^2+y^2=2x$
$x^2+y^2+2gx+2fy+c=0$
$C=(+1,0)$
$r=\sqrt{g^2+f^2-c}$
$=\sqrt{1^2+0-0}$
$r=1$



{We get 2 regions if strip is ∥ to x or y-axis}

$x^2+y^2-2x=0$
$x^2=0$ $(y^2=2x)$
$x=0$

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ii) Let take an elementary strip ll to y-axis.

	OAB	ABC
y-limits:-	$y=\sqrt{2x-x^2}$ to $y=\sqrt{2x}$	$y=x$ to $y=\sqrt{2x}$
x-limits	$x=0$ to $x=1$	$x=1$ to $x=2$

$$I = \int_0^1\int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2\int_x^{\sqrt{2x}} xy\,dy\,dx$$

$$= \int_0^1 x\,dx\int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy + \int_1^2 x\,dx\left[\frac{y^2}{2}\right]_x^{\sqrt{2x}}$$

$$= \int_0^1 x\,dx\left(\frac{y^2}{2}\right)_{\sqrt{2x-x^2}}^{\sqrt{2x}} + \int_1^2 x\,dx\left[\frac{(\sqrt{2x})^2}{2}-\frac{x^2}{2}\right]$$

$$= \int_0^1 x\,dx\left[\frac{(\sqrt{2x})^2}{2}-\frac{(\sqrt{2x-x^2})^2}{2}\right] + \int_1^2 x\,dx\left[\frac{2x}{2}-\frac{x^2}{2}\right]$$

$$= \frac{1}{2}\int_0^1 x\,dx\left\{2x-\left[2x+x^4-2\sqrt{2x}\times x\right]\right. + \frac{1}{2}\int_1^2 2x^2-x^3$$

$$= \frac{1}{2}\int_0^1 x\,dx\left\{\cancel{2x}-\cancel{2x}-x^4+2\sqrt{2}x\right\} + \frac{1}{2}\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)_1^2$$

$$= \frac{1}{2}\int_0^1 \cancel{2x^2}-x^5+2\sqrt{2}x^2 + \frac{1}{2}\left\{\frac{2\times 2^3}{3}-\frac{2\times 1^3}{3}\right.$$

$$= \frac{-1}{2}\left[\frac{x^6}{6} - \frac{2\sqrt{2}x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{14}{3} - \frac{15}{4}\right]$$

$$= \frac{-1}{2}\left[\frac{6}{6} - \frac{2\cancel{\sqrt{2}}\times 7}{3}\right] + \left[\frac{14}{6} - \frac{15}{8}\right]$$

$$= \frac{-1}{2}\left[\frac{3 - 2\cancel{\sqrt{2}}}{3}\right] + \left[\frac{56-45}{24}\right]$$

$$= \frac{-3}{6} + \frac{2\cancel{\sqrt{2}}}{\cancel{6}} + \frac{11}{24}$$

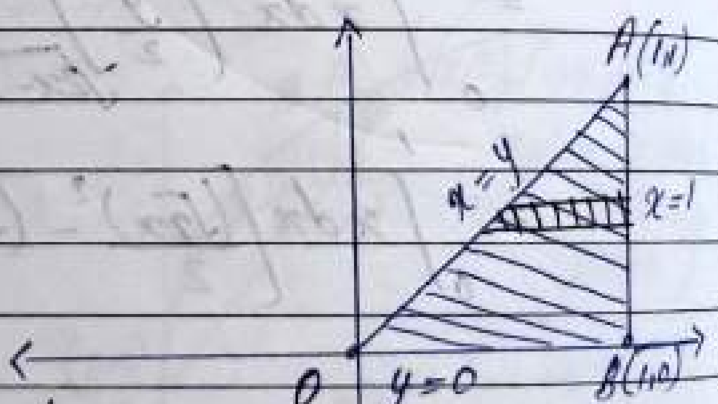
$$= \frac{11}{24} + \frac{2\cancel{\sqrt{2}}}{6} - \frac{3}{6}$$

$$= \frac{7}{12} \quad \underline{\underline{Ans}}$$

Ques (3) $I = \iint_R \frac{y e^{2y}}{\sqrt{(1-x)(1-y)}}\, dx\, dy$

where R is the region of Δ whose vertices are $(0,0)$ $(1,0)$ & $(1,1)$?

Sol.



This integral is very difficult to solve with y first and little easy to solve with x first. Therefore we take strip $\parallel$ to x-axis.

Take strip ∥ to x-axis

x-limits :- $x = y$ to $x = 1$

y-limits :- $y = 0$ to $y = 1$

$$I = \int_0^1 \int_y^1 \frac{y e^{2y}}{\sqrt{(1-x)(x-y)}}\, dx\, dy$$

$$= \int_0^1 y e^{2y} \int_y^1 \frac{1}{\sqrt{(1-x)(x-y)}}\, dx\, dy$$

Put $x - y = t^2$

$dx = 2t\,dt$

x	y	1
t	0	$\sqrt{1-y}$

$$I = \int_0^1 y e^{2y} \int_0^{\sqrt{1-y}} \frac{2\cancel{t}\,dt}{\sqrt{(1-y-t^2)\cancel{t^2}}}\, dy$$

$$= 2\int_0^1 y e^{2y} \int_0^{\sqrt{1-y}} \frac{dt}{\sqrt{(\sqrt{1-y})^2 - t^2}}\, dy$$

$$= 2\int_0^1 y e^{2y} \left[\sin^{-1}\left(\frac{t}{\sqrt{1-y}}\right)\right]_0^{\sqrt{1-y}}$$

$$= 2\int_0^1 y e^{2y} \left[\sin^{-1}(1)\right]$$

$$= 2\int_0^1 y e^{2y} \cdot \frac{\pi}{2}$$

$$= \pi \left[y \frac{e^{2y}}{2} - \frac{e^{2y}}{4}\right]_0^1$$

$$I = \frac{\pi}{4}(e^2+1)$$

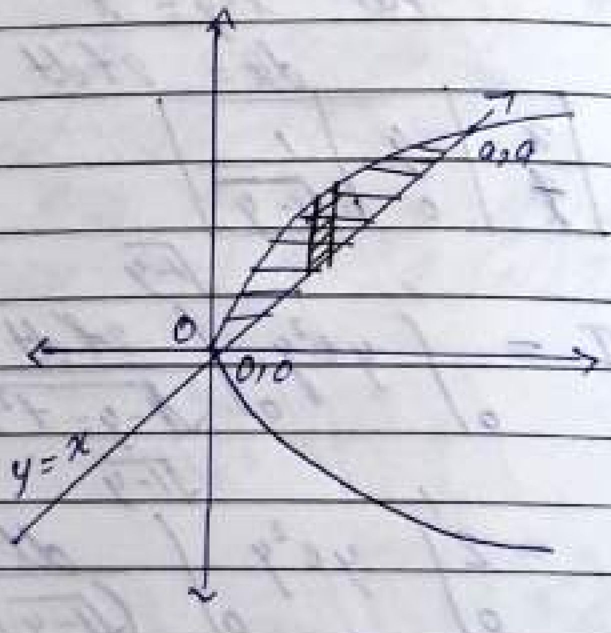
Ques (4) $\iint_R \frac{\tan^{-1}y}{(1+y^2)\sqrt{(1-x)(x-y)}}\,dx\,dy$ when R is Δ bounded by (0,0) (1,1), (1,0) ?

Ques (5) Evaluate :- $\iint_R \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$

$\int y$ is easy

R is region bounded by $y^2 = ax$, $y = x$.

Sol.



x limits :- $x = 0$ to $x = a$

y limits :- $y = x$ to $y = \sqrt{ax}$

$$I = \int_0^a \int_{y=x}^{\sqrt{ax}} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^a \left[\int_a^{\sqrt{ax}} \frac{y\,dy}{\sqrt{ax-y^2}} \right] \frac{dx}{(a-x)}$$

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$$I = \int_0^a \frac{dx}{(a-x)} \left(-\sqrt{ax-y^2}\right)_x^{\sqrt{ax}}$$

$$= \int_0^a \frac{dx}{(a-x)} \left(\sqrt{ax-x^2}\right) \quad \left\{ \sqrt{\cancel{ax} - (\cancel{\sqrt{ax}})^2} \right\}$$

$$= \int_0^a \frac{dx\, \sqrt{x}\left(\sqrt{a-x}\right)}{(a-x)} = \int_0^a \frac{\sqrt{x}\, dx}{\sqrt{a-x}}$$

$$= \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}} dx = \int_0^a \frac{x^{1/2}}{(a-x)^{1/2}} dx$$

Put $x = at$

$dx = a\,dt$

a	0	a
t	0	1

$$\therefore I = \int_0^1 \frac{(at)^{1/2}}{(a-at)^{1/2}} a\,dt$$

$$= a^{1/2+1-1/2} \int_0^1 \frac{t^{1/2}}{(1-t)^{1/2}} dt$$

$$= a \int_0^1 t^{3/2-1} (1-t)^{-1/2} dt$$

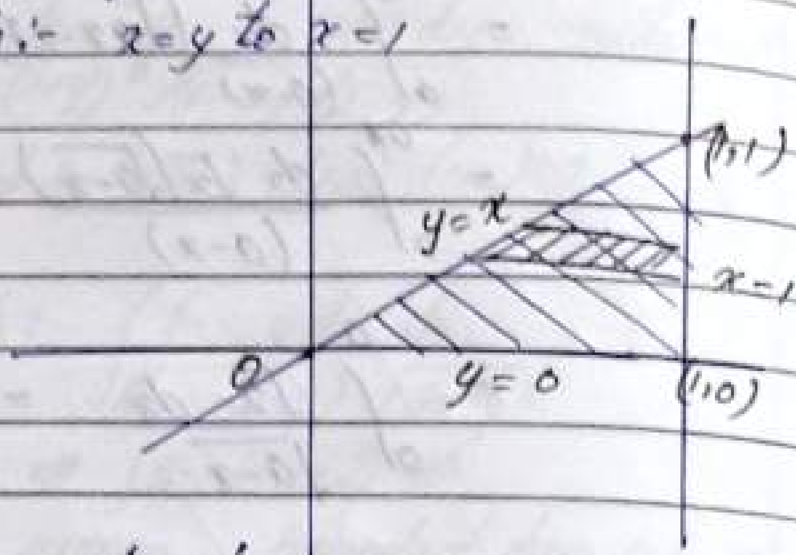
$$= a\, \beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= a \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma 2} = a \frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}$$

$$I = \frac{\pi a}{2}$$ Ans

Ques (4) y-limits : $y=0$ to $y=1$.

x-limits :- $x=y$ to $x=1$



$$\therefore I = \int_{y=0}^{1} \int_{x=y}^{1} \frac{\tan^{-1} y}{(1+y^2)\sqrt{(1-x)(1-y)}} \, dx \, dy$$

Put $x-y = (1-y)t$

$x = y + (1-y)t$

$dx = (1-y)dt$

x	y	1
t	0	1

$1-x = (1-y) - (1-y)t$

$= (1-y)(1-t)$

$$\therefore I = \int_{0}^{1} \int_{t=0}^{1} \frac{\tan^{-1} y}{(1+y^2)} \left[(1-y)^{-1/2} t^{-1/2} (1-y)^{-1/2} (1-t)^{-1/2} (1-y) \, dt \, dy \right]$$

$$= \int_0^1 \frac{\tan^{-1} y}{1+y^2} \, dy \int_0^1 t^{-1/2} (1-t)^{-1/2} \, dt$$

$$= \left[\frac{(\tan^{-1} y)^2}{2} \right]_0^1 \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{\pi^2}{32} \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1} = \frac{\pi^3}{32}$$

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Ans (7) Evaluate: $\int_0^1\int_0^x x(x^2+y^2)\,dy\,dx$

$$I = \int_0^1\int_0^x x(x^2+y^2)\,dy\,dx$$

$$= \int_0^1 x\,dx\left[x^2y+\frac{y^3}{3}\right]_0^x$$

$$= \int_0^1 x\,dx\left[\frac{3x^2y+y^3}{3}\right]_0^x$$

$$= \int_0^1 \frac{x\,dx}{3}\left(3x^3+x^3\right)$$

$$= \frac{1}{3}\int_0^1 (3x^4+x^4)\,dx$$

$$= \frac{1}{3}\left[\frac{3x^5}{5}+\frac{x^5}{5}\right]_0^1$$

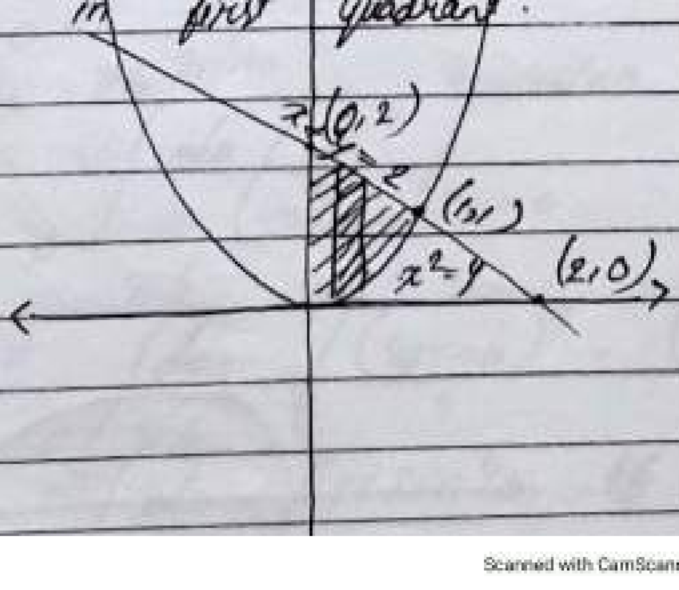
$$I = \frac{4}{15} \quad \underline{\underline{\text{Ans}}}$$

~~First~~

Ans (8) Find the integral

$$I = \iint_R y\,dy\,dx$$ over the region

R where R is bounded by $x=0$, $x^2=y$, $x+y=2$ in first quadrant.



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$$I = \int_{0}^{1}\int_{y=x^2}^{y=2-x} y \, dy \, dx$$

$$= \int_{0}^{1} dx \int_{y=x^2}^{y=2-x} y \, dy$$

$$= \int_{0}^{1} dx \left[\frac{y^2}{2}\right]_{x^2}^{2-x}$$

$$= \int_{0}^{1} dx \left[\frac{(2-x)^2}{2} - \frac{(x^2)^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{(2-x)^3}{-3} - \frac{x^5}{5}\right]_{0}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}(8-1) - \frac{1}{5}\right]$$

$$= \frac{1}{3}\left[\frac{7}{3} - \frac{1}{5}\right]$$

$$= \frac{16}{15} \text{ Ans}$$

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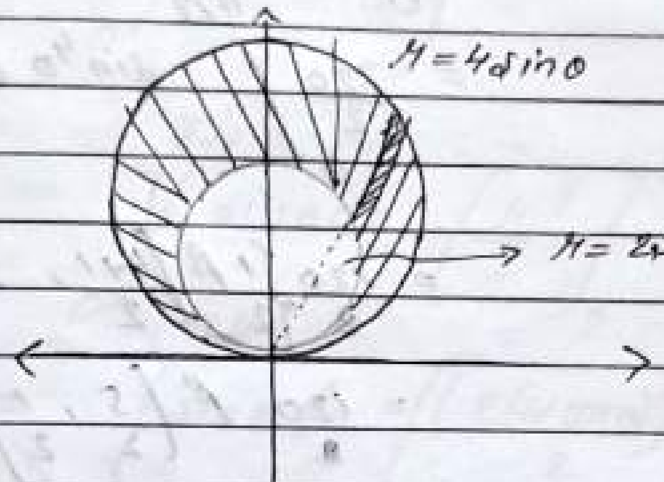
Double Integration Using Polar Form:

Que① $\iint r^3\,dr\,d\theta$ where the region are bounded by $r = 2\sin\theta$ and $r = 4\sin\theta$

Sol.



Consider I-quadrant
r-limit:- $r=2\sin\theta$ to $r=4\sin\theta$
θ-limits:- $\theta=0$ to $\theta=\pi/2$
(Takes twice due to symm)

$$I = \int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3\,dr$$

$$= \int_0^{\pi/2} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= \int_0^{\pi/2} d\theta \left[\frac{(4\sin\theta)^4}{4} - \frac{(2\sin\theta)^4}{4}\right]$$

$$= \int_0^{\pi/2} d\theta \cdot \left[\frac{256\sin^4\theta}{4} - \frac{16\sin^4\theta}{4}\right]$$

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$$= \int_0^{\pi/2} d\theta \left[\frac{\cancel{80}\sin^4\theta}{4}\right]$$

$$= \int_0^{\pi/2} \frac{120\sin^4\theta}{2}\, d\theta$$

$$= \frac{120}{2}\int_0^{\pi/2} \sin^4\theta\, d\theta$$

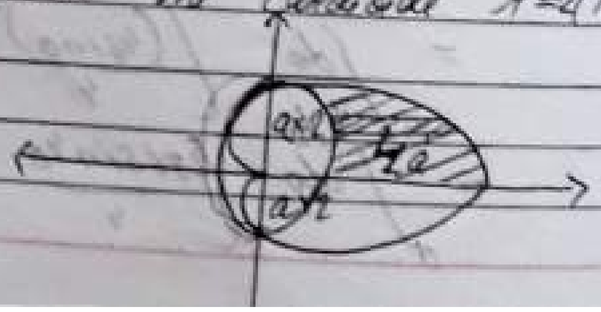
$$= \frac{120}{2}\left\{\frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{0+1}{2}\right)\right\}$$

$$= \frac{120}{4}\,\beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= \frac{120}{4}\,\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{6}{2}} =$$

$$I = 60\sqrt{\pi}\,\frac{\Gamma 5/2}{\Gamma 3}$$

Ques ② Evaluate $\iint_R \sin\theta \; r\,dr\,d\theta$ where R is the area in the first quadrant outside the circle $r=2$ and inside the cardioide $r=2(1+\cos\theta)$



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$$I = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta \int_2^{2(1+\cos\theta)} r\,dr$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)}$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta \left[\frac{[2(1+\cos\theta)]^2}{2} - \frac{(2)^2}{2}\right]$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta \left[\frac{4(1+2\cos\theta+\cos^2\theta)}{2} - \frac{4}{2}\right]$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta \left[2(1+2\cos\theta+\cos^2\theta)-2\right]$$

$$= 2\int_0^{\pi/2} \sin\theta\,d\theta\,(1+2\cos\theta+\cos^2\theta - 1)$$

$$= 2\int_0^{\pi/2} 2\sin\theta\cos\theta\,d\theta + \int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta$$

$$= \cancel{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} + \int_0^{\pi/2} \sin\theta(1-\sin^2\theta)\,d\theta$$

$$= \cancel{2}\left[\cancel{\frac{-\cos\pi}{2} + \frac{\cos 0}{2}}\right] + \int_0^{\pi/2} [\sin\theta - \sin^3\theta]\,d\theta$$

$$= -2\left[\cancel{\frac{-1}{2} + \frac{1}{2}}\right]$$

$$= 4\left[\frac{1}{2}\cdot\beta\left(\frac{1+1}{2}, \frac{1+1}{2}\right)\right]$$

$$= 2\ \beta(1,1) + \left[\frac{-\cos 3\theta}{-3} - \cos\theta\right]_0^{\pi/2}$$

$$= 2\beta(1,1) + \left[-\left\{\frac{\cos 3\theta}{3} + \cos\theta\right\}\right]_0^{\pi/2}$$

$$= 2\beta(1,1) - \left[\frac{\cos 3\pi/2}{3} - \frac{\cos 3\times 0}{3} + \cos\pi/2 - \cos 0\right]$$

$$= 2\beta(1,1) - [0 - 1 + 1 - 1]$$

$$= 2\beta(1,1) + \frac{1}{2}\beta\left[1, \frac{3}{2}\right]$$

$$= 2\beta(1,1) + \frac{1}{2}\left[\frac{\sqrt{1}\,\sqrt{3/2}}{\sqrt{5/2}}\right]$$

$$= 2\frac{\sqrt{1}\sqrt{1}}{\sqrt{2}} + \frac{1}{2}\frac{\sqrt{1}\ \frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}}\,\frac{3}{2}}$$

$$= \frac{2\sqrt{1}\sqrt{1}}{\sqrt{2}} + \frac{1}{2}\frac{\sqrt{1}\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt{\frac{1}{2}}}$$

$$= \frac{2\sqrt{1}}{\sqrt{2}} + \frac{1}{2}\frac{\sqrt{1}}{1/2}$$

$$I = \frac{8}{3}$$

Date 05/04/17

Change the order of Integration

Que 1

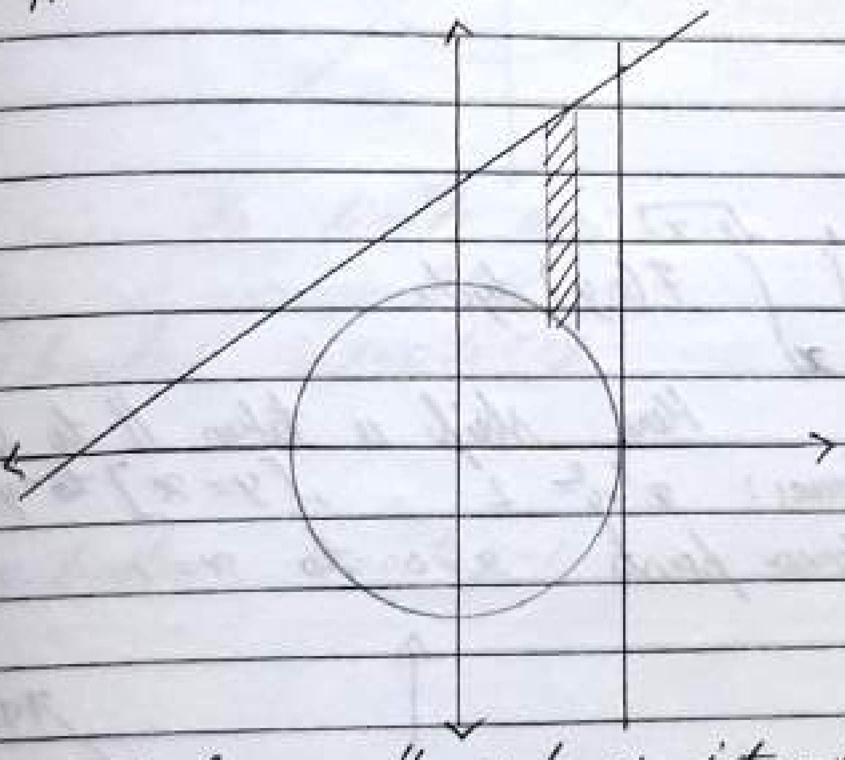
$$I = \int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} F(x,y)\, dy\, dx$$

Sol. Here, strip is taken ∥ to y-axis from $y = \sqrt{a^2-x^2}$ to $y = x+3a$

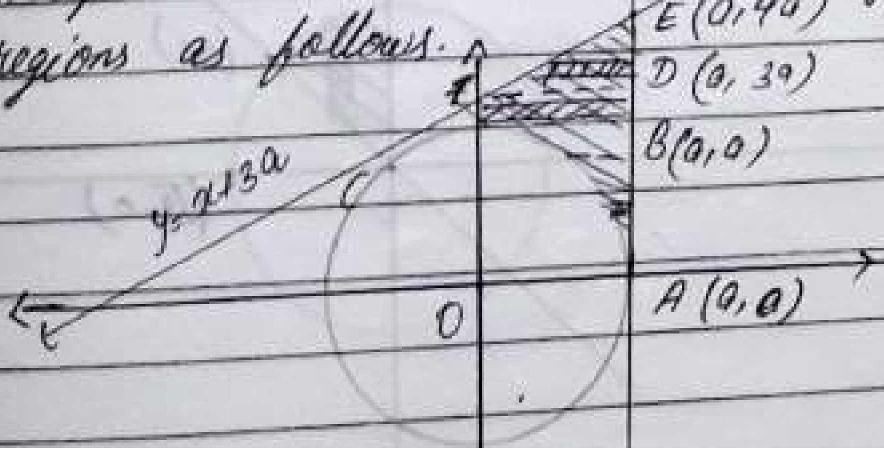
Curves: $x^2+y^2 = a^2$

and x varies from

corner pts $x = 0$ to $x = a$.



To change the order of integration we take strip ∥el to x-axis. Here, we get three regions as follows.



$\therefore$ The limits are :-

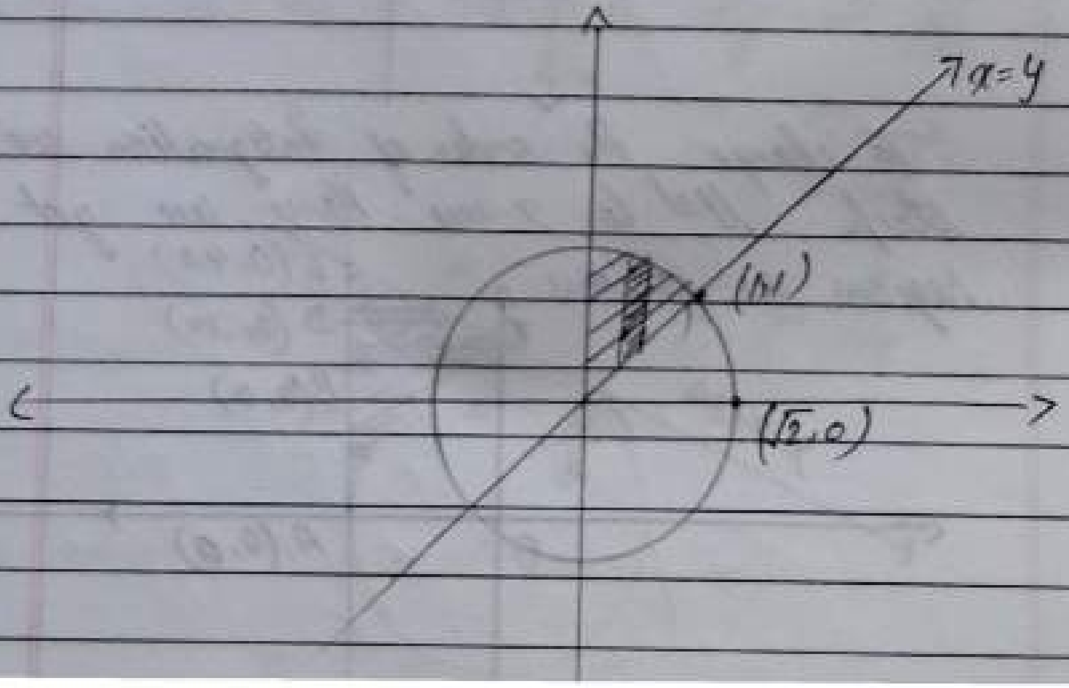
x limits	ABC	BCDF	DEF
	$x = \sqrt{a^2 - y^2}$ to $x = a$	$x = 0$ to $x = a$	$x = y - 3a$ to $x = a$
y limits	$y = 0$ to $y = a$	$y = 0$ to $3a$	$y = 3a$ to $y = 4a$

Ques ② $I = \int_0^1 \int_x^{\sqrt{2 - x^2}} F(x,y)\, dy\, dx$

Here strip is taken ∥ to y-axis

Curves: $x^2 + y^2 = 2$, $[y = x]$ to $y = \sqrt{2 - x^2}$

corner points $x = 0$ to $x = 1$



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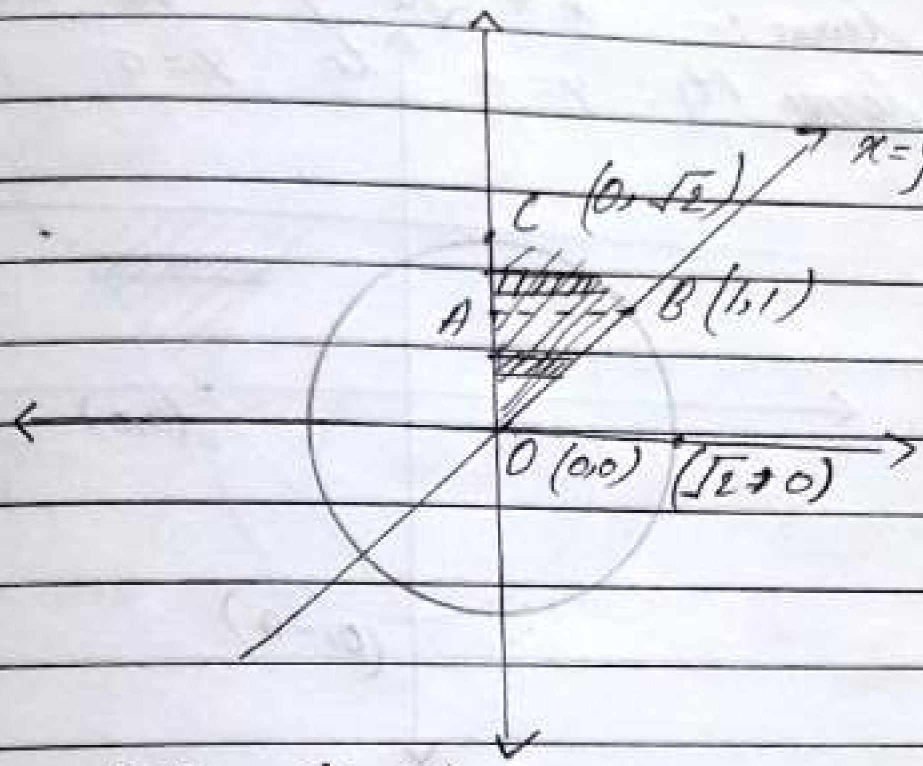
To change the order of integration we take strip ||el to x-axis.



$\therefore$ The limits are:

	ABO	ABC
x limits:	$x=0$ to $x=y$	$x=0$ to $x=\sqrt{2-y^2}$
y-limits	$y=0$ to $y=1$	$y=1$ to $y=\sqrt{2}$

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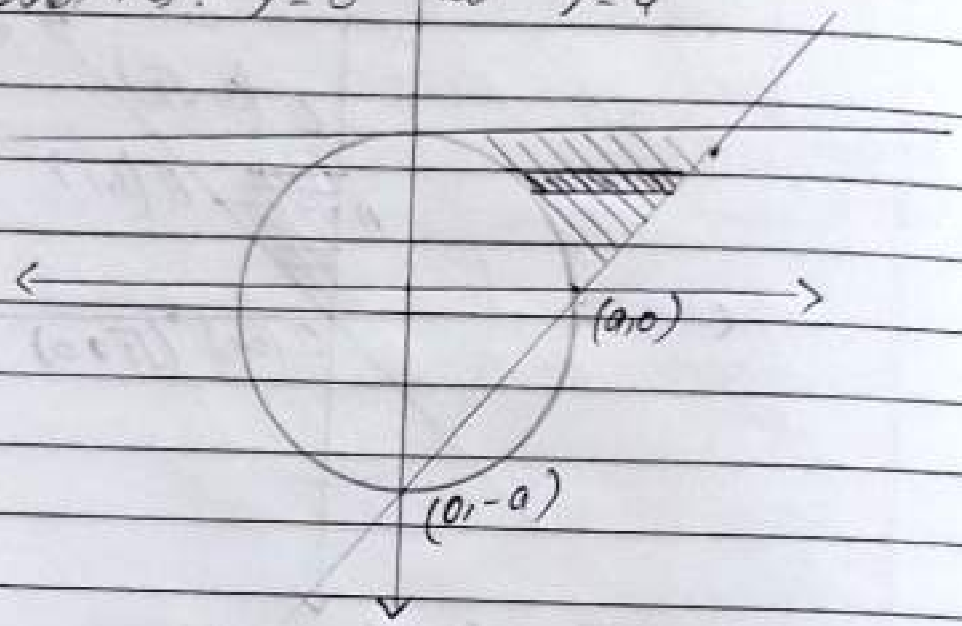
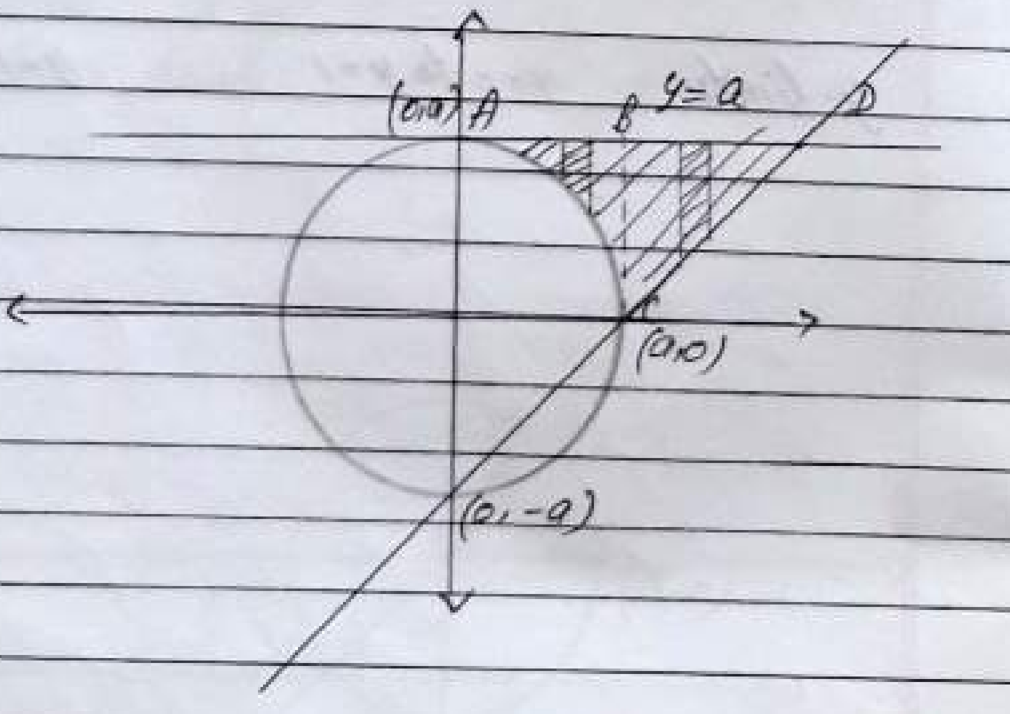
Ques ③ $I = \int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\, dx\, dy$

Curve :- $x = \sqrt{a^2-y^2}$ to $x = (y+a)$

Corner Pts : $y = 0$ to $y = a$



To change the ~~integration~~ order of integration we take strip ∥ to y-axis.



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The limits are:

	ABC	BCD
x-limits	$x=0$ to $x=a$	$x=a$ to $x=2a$.
y-limits	$y=a$ to $y=\sqrt{a^2-x^2}$	$y=a$ to $y=x-a$

$$I = \int_{x=0}^{x=a} \int_{y=\sqrt{a^2-x^2}}^{y=a} f(x,y)\,dy\,dx + \int \int_{y=x-a}^{y=a} f(x,y)\,dy\,dx$$

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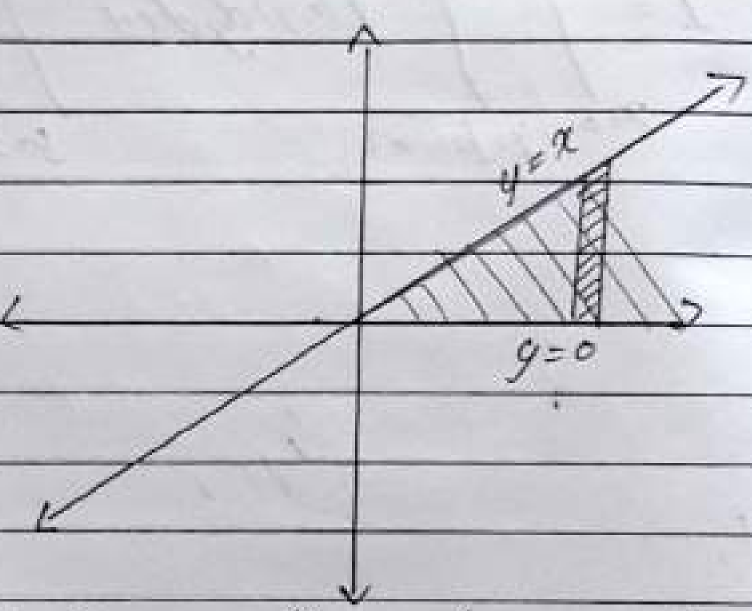
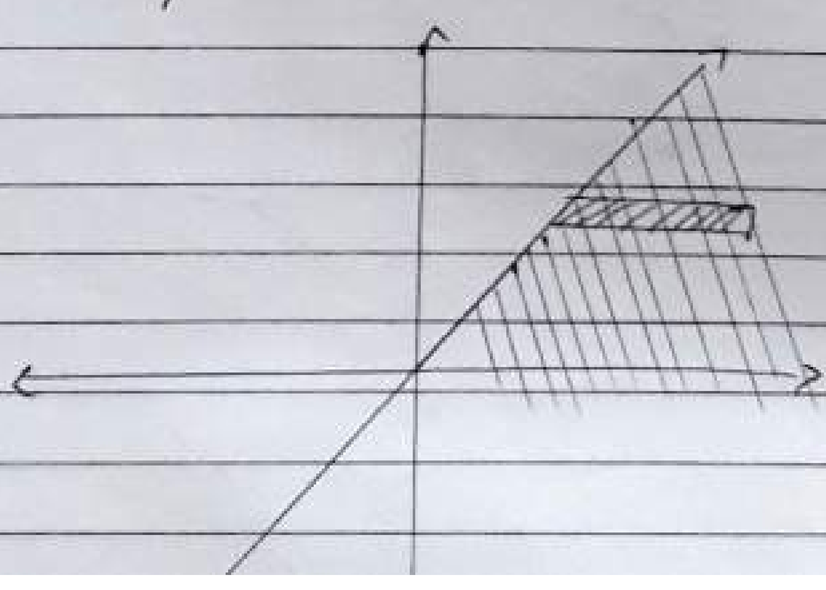
Change the Order of Integration & Evaluate :-

Ques 9 $$I = \int_0^{\infty}\int_0^{x} x\, e^{-x^2/y}\, dy\, dx$$

Curves :- $y = 0$ to $y = x$
Corner Pts :- $x = 0$ to $x = \infty$



To change the order of integration we take strip $\parallel^{el}$ to x-axis.



Limits are:-

x- limits:- $x = y$ to $x = \infty$

y- limits:- $y = 0$ to $y = \infty$

$$I = \int_0^\infty \int_y^\infty x \cdot e^{-x^2/y}\, dx\, dy$$

$$= \int_0^\infty \frac{-y}{2} \int_y^\infty -\frac{2}{y} x \cdot e^{-x^2/y}\, dx\, dy$$

$$= \int_0^\infty -\frac{y}{2}\, dy \left[e^{-x^2/y}\right]_y^\infty$$

$$= \int_0^\infty -\frac{y}{2} \left[e^{-\infty^2/y} - e^{-y^2/y}\right]$$

$$= \int_0^\infty -\frac{y}{2} \left[-e^{-y}\right] dy$$

$$= \frac{1}{2} \int_0^\infty y e^{-y} dy$$

$$= \frac{1}{2} \int_0^\infty y^{2-1} e^{-y} dy$$

$$= \frac{1}{2} \sqrt{2} = \frac{1!}{2}$$

$$I = \frac{1}{2}$$ Ans

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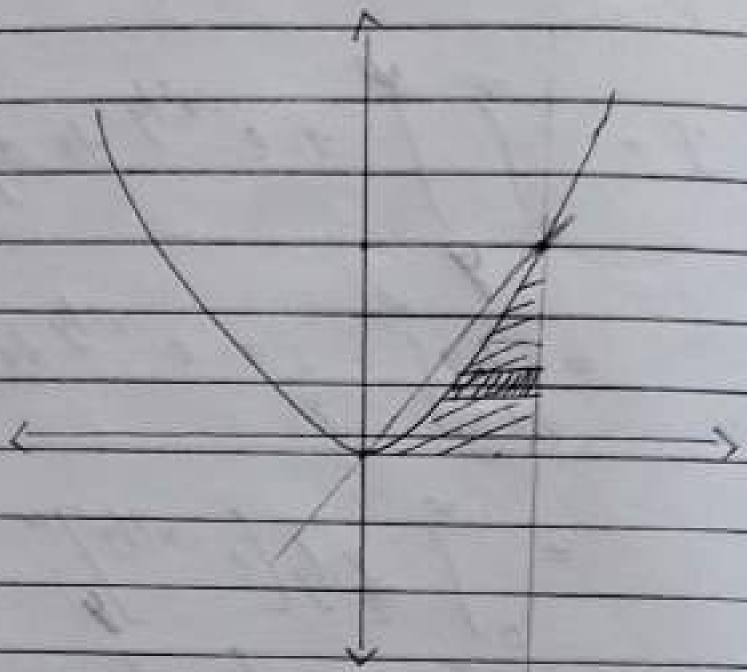
#(16, 17)#

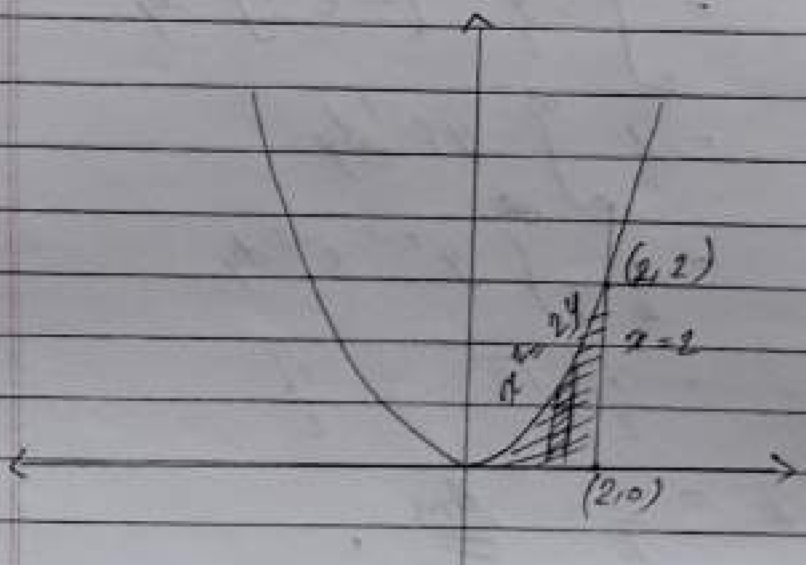
Ques 2 $\displaystyle\int_{0}^{2}\int_{\sqrt{2y}}^{2} \frac{x^2}{\sqrt{x^4-4y^2}}\,dx\,dy$

Sol Curves: $x=\sqrt{2y}$, $x=2$

Corner pts :- $y=0$ to $y=2$



To change the order of integration we take strip ||el to y-axis.



x limits :- $x=0$ to $x=2$

y- limits $y=0$ to $y=x^2/2$

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$$I = \int_0^2 \int_{y=0}^{y=x^2/2} \frac{x^2}{\sqrt{x^4-4y^2}}\, dy\, dx$$

$$I = \int_0^2 x^2 \int_{y=0}^{x^2/2} \frac{1}{\sqrt{x^4-4y^2}}\, dy\, dx$$

$$= \frac{1}{2}\int_0^2 x^2 \int_0^{x^2/2} \frac{1}{\sqrt{\left(\frac{x^2}{2}\right)^2 - y^2}}\, dy\, dx$$

$$= \frac{1}{2}\int_0^2 x^2 \left[\sin^{-1}\left(\frac{y}{x^2/2}\right)\right]_0^{x^2/2} dx$$

$$= \frac{1}{2}\int_0^2 x^2 \left[\sin^{-1}\left(\frac{x^2/2}{x^2/2}\right) - \sin^{-1}\left(\frac{0}{x^2/2}\right)\right]$$

$$= \frac{1}{2}\int_0^2 x^2 \frac{\pi}{2}$$

$$= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$I = \frac{\pi}{4}\,\frac{8}{3} = \frac{2\pi}{3} \quad \text{Ans}$$

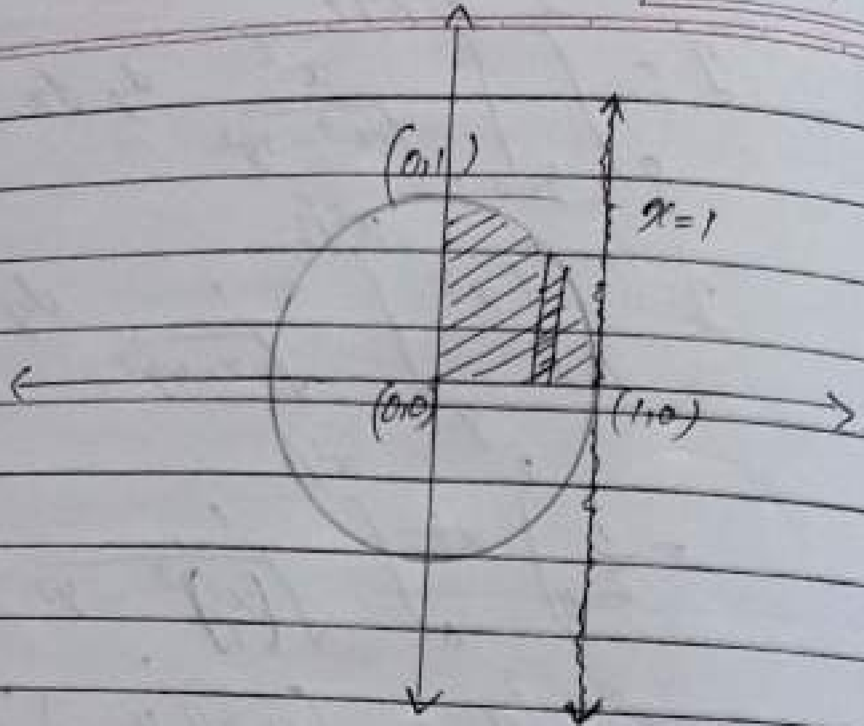
Que 13 $$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dy\, dx$$

Curve: $y=0$ to $y=\sqrt{1-x^2}$

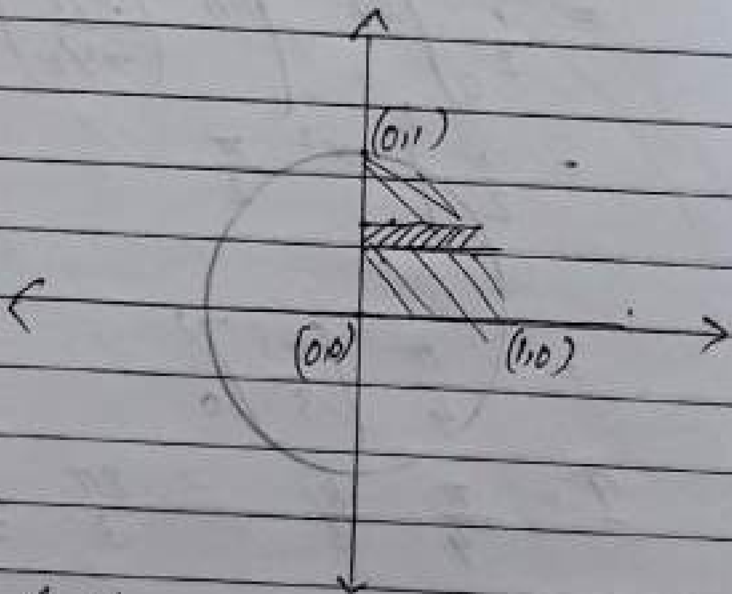
Corner pts:- $x=0$ to $x=1$.

$y^2 = 1-x^2$

$x^2+y^2=1$



To change the order of integration we take strip ||el to x-axis



x-limits :- $x=0$ to $x=\sqrt{1-y^2}$
y-limits :- $y=0$ to $y=1$

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} \, dx \, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)} \frac{1}{\sqrt{1-y^2-x^2}} \, dx \, dy$$

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$$= \int_0^1 \frac{e^y}{e^y+1}\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}}$$

$$= \int_0^1 \frac{e^y}{e^y+1}\,\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left[\log(e^y+1)\right]_0^1$$

$$= \frac{\pi}{2}\left[\log(e^1+1) - \log(e^0+1)\right]$$

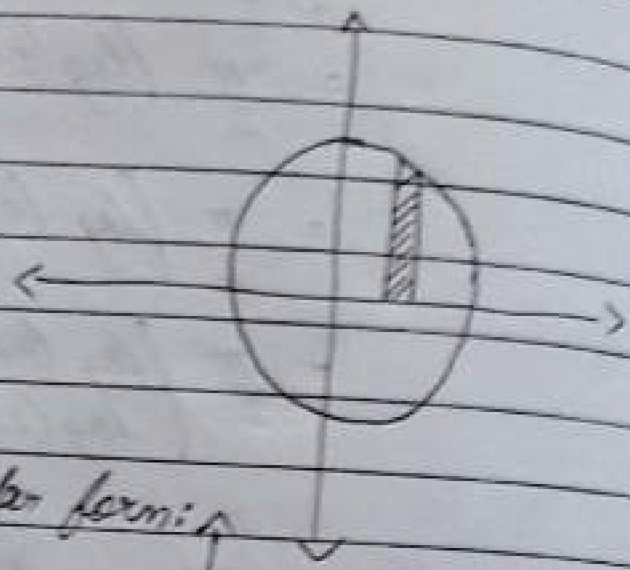
$$= \frac{\pi}{2}\left[\frac{\log(e+1)}{\log(2)}\right] \quad \underline{\underline{\text{Ans}}}$$

Exam (1) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$

$x = 0, \; x = a.$



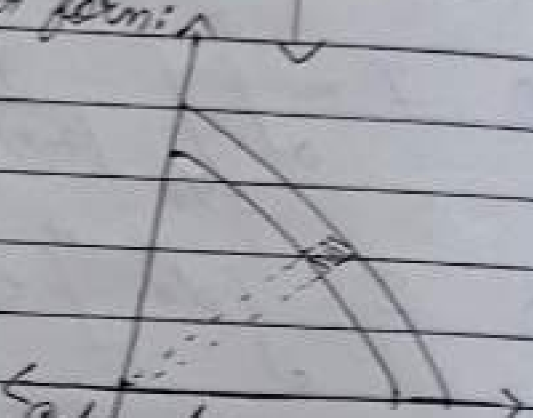
Converting to Polar form:

Put $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$



Eqn of curves in Polar form

$r^2 = a^2 \qquad (\because x^2+y^2 = r^2)$

$r = a \qquad r = 0$ to $r = a$

$\theta = 0$ to $\theta = \pi/2$

$$I = \int_0^{\pi/2}\int_0^{a} r^2 \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} r^3\,dr\,d\theta$$

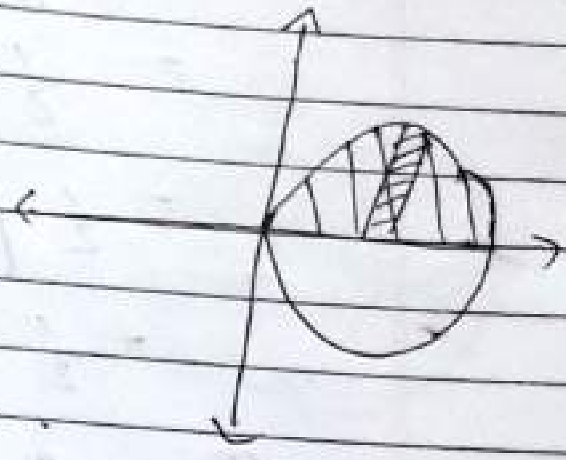
$$= \frac{1}{4}\int_0^{\pi/2} \left[r^4\right]_0^a d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} a^4\,d\theta$$

$$= \frac{a^4}{4}\left[\theta\right]_0^{\pi/2} = \frac{\pi a^4}{8} \quad \text{Ans}$$

Ques 2

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$



$y^2 = 2x - x^2$

$x^2+y^2-2x=0$

$c=(1,0)$

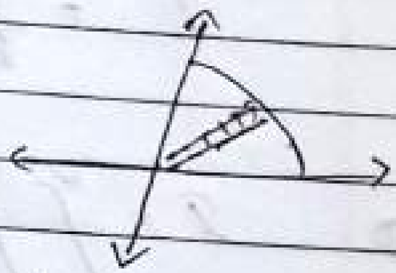
$r=\sqrt{g^2+f^2-c}$

$=1$

Put $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$



$dx\,dy = r\,dr\,d\theta$

Eqn of curves in Polar form:

$r^2=a^2$

$r=a \qquad r^2 = 2r\cos\theta$

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta}{\sqrt{r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cancel{\cos^2\theta\, d\theta}\, [4\cos^3\theta]\, d\theta$$

$$= \cancel{\frac{2}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_0^{\pi/2}}$$

$$= \cancel{\left[\frac{0+\pi}{2}+0\right]}$$

$$= \cancel{\pi}$$

$$I = \frac{2}{4} \int_0^{\pi/2} (\cos 3\theta + 3\cos\theta)\, d\theta$$

$$= \frac{1}{2} \left[\frac{\sin 3\theta}{3} + 3\sin\theta \right]_0^{\pi/2}$$

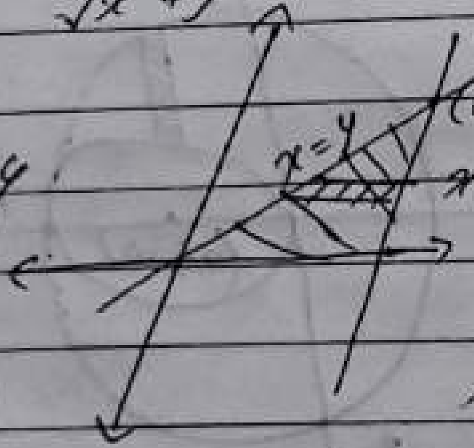
$$= \frac{1}{2} \left[\frac{\sin 3\pi/2}{3} - \frac{\sin 0}{3} \right] + \left[3\sin\pi/2 - \ldots \right]$$

$$= \frac{1}{2} \left\{ \left[\frac{-1}{3} - 0 \right] + \left[3 - 0 \right] \right\}$$

$$I = \frac{8}{3 \cdot 2} = \frac{8}{6} = \frac{4}{3} \text{ Ans}$$

Ques (5) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \, dx\, dy$

$x = a, \quad x = y$



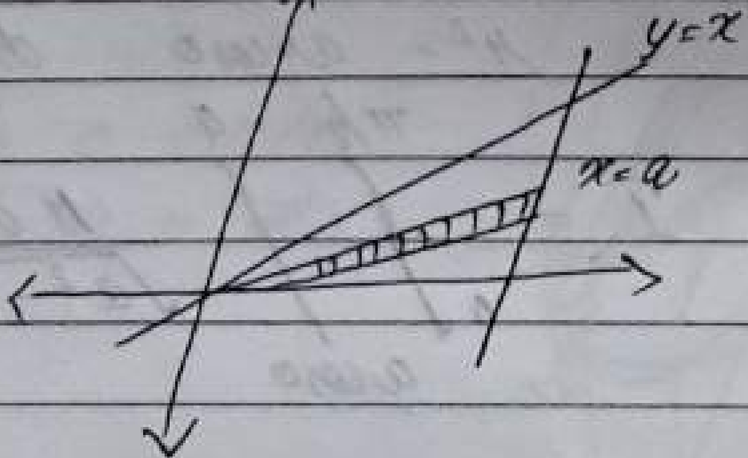
$a = r\cos\theta$

$r = \frac{a}{\cos\theta}$

$r\cos\theta = r\sin\theta$

$1 = \tan\theta$

$\theta = \pi/4$



$$I = \int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2\cos^2\theta \; r\,dr\,d\theta}{\sqrt{r^2}}$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta \left[r^3\right]_0^{a/\cos\theta}$$

$$\frac{1}{3}\int_0^{\pi/4} \cos^2\theta \times \frac{a^3}{\cos^3\theta}\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \frac{a^3}{\cos\theta}\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \frac{a^3}{3}\left[\log|\sec\theta + \tan\theta|\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log(\sqrt{2}+1) - \log(1+0)\right]$$

$$= \frac{a^3}{3}\log\left(1+\sqrt{2}\right) \text{ Ans}$$

09/01

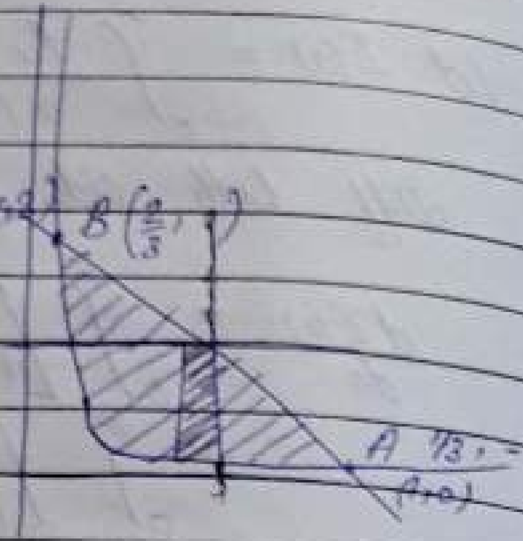
Exam (5) Find the area bounded by the curves $9xy = 4$ and $2x+y = 2$.

$y = \frac{4}{9x}$

$2x+y=2$

$x = 1$

$y = 2$



To find co-ordinates of A and B:-

$$2x + \frac{4}{9x} = 2$$

$$18x^2 + 4 = 18x$$

$$18x^2 - 18x + 4 = 0$$

$$9x^2 - 9x + 2 = 0$$

$$x = 1/3 , 2/3$$

Now, $y = \frac{4}{9x}$ to $y = 2-2x$

x-limits, $x = \frac{1}{3}$ to $x = \frac{2}{3}$

$$A = \int_{1/3}^{2/3}\int_{4/9x}^{2-2x} dy\,dx$$

$$= \int_{1/3}^{2/3} \left[y\right]_{4/9x}^{2-2x} dx$$

$$= \int_{1/3}^{2/3} \left(2-2x - 4/9x\right) dx$$

$$= \int_{1/3}^{2/3} \left[\frac{18x - 18x^2 - 4}{9x}\right] dx$$

$$= \frac{1}{9}\int_{1/3}^{2/3} \left[\frac{18x - 18x^2 - 4}{x}\right]$$

$$= \frac{1}{9}\left[18x - \frac{18x^2}{2} - 4\log x\right]_{1/3}^{2/3}$$

$$= \frac{1}{9}\left[\left(\frac{18 \times 2}{3} - \frac{18 \times 1}{3}\right) - \left(\frac{9 \times 4}{9} - \frac{9 \times 1}{9}\right)\right.$$

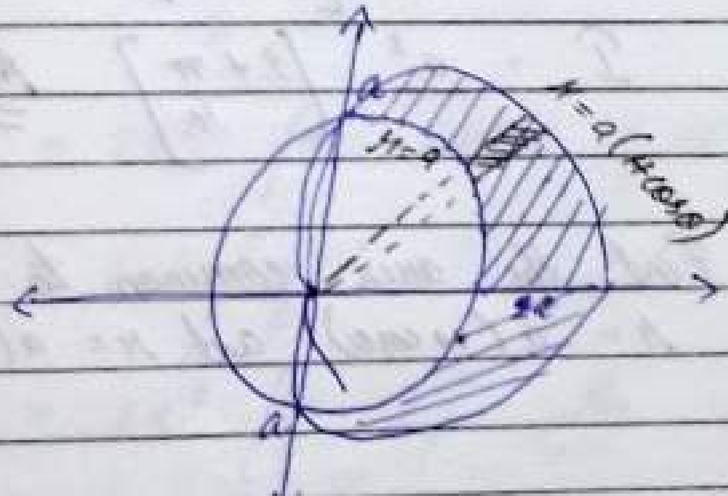
$$\left. - 4\left[\log\frac{2}{3} - \log\frac{1}{3}\right]\right]$$

$$\frac{1}{9}\left[(12 - 6) - 3 - 4\log 2\right]$$

$$= \frac{1}{9} \times \left[6 - 3 - 4\log 2\right]$$

$$I = \left[\frac{1}{3} - \frac{4\log 2}{9}\right] \text{ Ans}$$

Ques (15) Find the area outside the circle $r = a$ and inside the cardioide $r = a(1+\cos\theta)$



$r = a$ to $r = a(1+\cos\theta)$

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$\theta = 0$ to $\theta = \pi/2$

$$I = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{a^2(1+\cos^2\theta+2\cos\theta)}{2} - \frac{a^2}{2}\right]$$

$$= \frac{2a^2}{2}\int_0^{\pi/2}\left[1+\cos^2\theta+2\cos\theta-1\right]d\theta$$

$$= a^2\int_0^{\pi/2}\cos^2\theta+2\cos\theta\,]\,d\theta$$

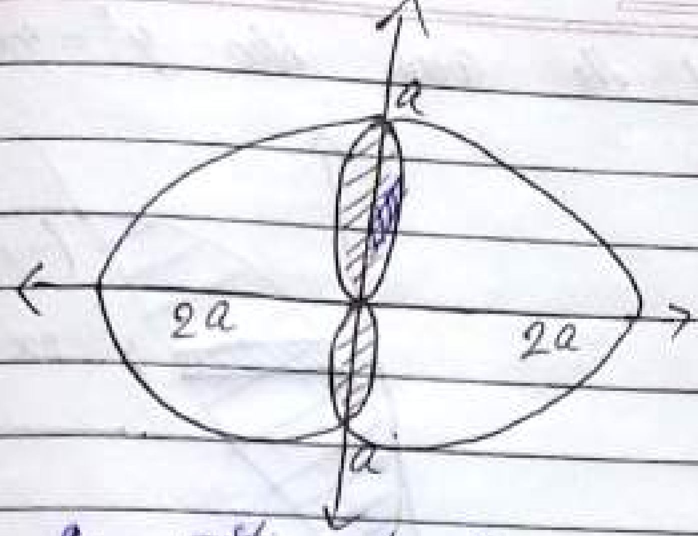
$$= a^2\int_0^{\pi/2}\left[\frac{1+\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi/2}\left[\frac{1}{2}+\frac{\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta\right]_0^{\pi/2}$$

$$I = \frac{a^2}{}\left[2+\frac{\pi}{4}\right] \quad \underline{\underline{\text{Ans}}}$$

Ques Find the area common to cardioide $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$



In I^{st} quadrant:

$r = 0$ to $r = a(1-\cos\theta)$

$\theta = 0$ to $\theta = \pi/2$:

$$A = 4\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left[r^2\right]_0^{a(1-\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} \left[a^2(1-2\cos\theta+\cos^2\theta)-0\right]$$

$$= 2a^2\int_0^{\pi/2} (1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 2a^2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2a^2\left[\frac{\pi}{2} - 2 + \frac{\pi}{4}\right]$$

$$= 2a^2\left[\frac{3\pi}{4} - 2\right]$$ Ans

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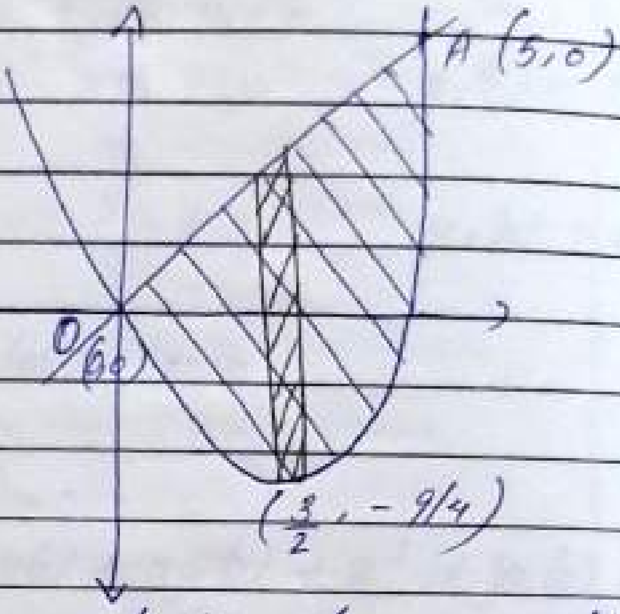
Mass

Ques (17) A lamina is bounded by curves $y = x^2 - 3x$ and $y = 2x$. If the density at any pts is given by $\rho(x,y) = \frac{24xy}{25}$.

(17) Prove that the mass is 175 units.

(17)

Sol.



$y = x^2 - 3x$

$y + \frac{9}{4} = x^2 - 3x + \frac{9}{4}$

$\left(y + \frac{9}{4}\right) = \left(x - \frac{3}{2}\right)^2$

$\Rightarrow (x-h) = 4a(y-k)$

Vertex: $\left(\frac{3}{2}, -9/4\right)$

Wherever quadratic terms are there make them perfect square.

$$I = \int_0^5 \int_{x^2-3x}^{2x} \frac{24}{25} xy \, dy \, dx$$

$$= \frac{24}{25} \int_0^5 \int_{x^2-3x}^{2x} xy \, dy \, dx$$

$$= \frac{24}{25} \int_0^5 x \left[\frac{y^2}{2}\right]_{x^2-3x}^{2x}$$

$$= \frac{24}{25 \times 2} \int_0^5 x \left[4x^2 - (x^2-3x)^2\right] dx$$

$$= \frac{12}{25} \int_0^5 \left[4x^3 - x^5 + 6x^4 - 9x^3\right] dx$$

$$= \frac{12}{25}\left\{\left[\frac{-5x^4}{4} - \frac{x^6}{6} + \frac{6x^5}{5}\right]_0^5\right.$$

$$= \frac{12}{25}\left[\left[\frac{-5\times5^4}{4} - \frac{5^6}{6} - \frac{6\times5^5}{5}\right] - 0\right]$$

$$= \frac{12}{25}\left[-781.25 - 2604.16 + 3750\right]$$

$$= \frac{12}{25} \times 364.59$$

$$= 175.008$$

$I \approx 175$ units

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Triple Integration

Ques (1) Evaluate $\iiint \frac{1}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz$

over the volume b/w a spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ $(b>a)$.

Sol. Converting to spherical polar co-ordinates

Put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$x^2+y^2+z^2 = r^2$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\therefore$ Eqn. of sphere become

$r^2 = a^2$

$r = a$

& $r = b$

In I^{st} octant:

$r = a$ to $r = b$

$\theta = 0$ to $\theta = \pi/2$

$\phi = 0$ to $\phi = \pi/2$

$$I = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_a^b \frac{1}{(r^2)^{3/2}}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta \int_a^b \frac{1}{r}\,dr$$

$$= 8\,[\phi]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\,[\log r]_a^b$$

$$= 8\,\frac{\pi}{2}\cdot 1\,[\log b - \log a]$$

$$= 4\pi\,\log\left(\frac{b}{a}\right)$$

Scanned with CamScanner

Ques ② Evaluate $\iiint \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\,dx\,dy\,dz$

Sol. Converting into spherical polar form.

Let $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$x^2+y^2+z^2 = r^2$

$\therefore$ Eqn. of sphere becomes

$r = a$

$r = 0$ to $r = a$

$\theta = 0$ to $\theta = \pi/2$

$\phi = 0$ to $\phi = \pi/2$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{a} \frac{r^2}{\sqrt{a^2-r^2}}\,dr$$

$$= [\phi]_0^{\pi/2}\,(-\cos\theta)_0^{\pi/2} \int_0^{a} r^2 (a^2-r^2)^{-1/2}\,dr$$

$= \frac{\pi}{2}\cdot 1$ (Put $r^2 = a^2\sin^2\beta$)

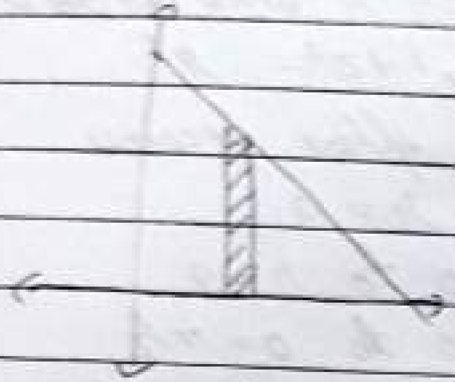
Volume:

$$V = \iiint dx\,dy\,dz$$

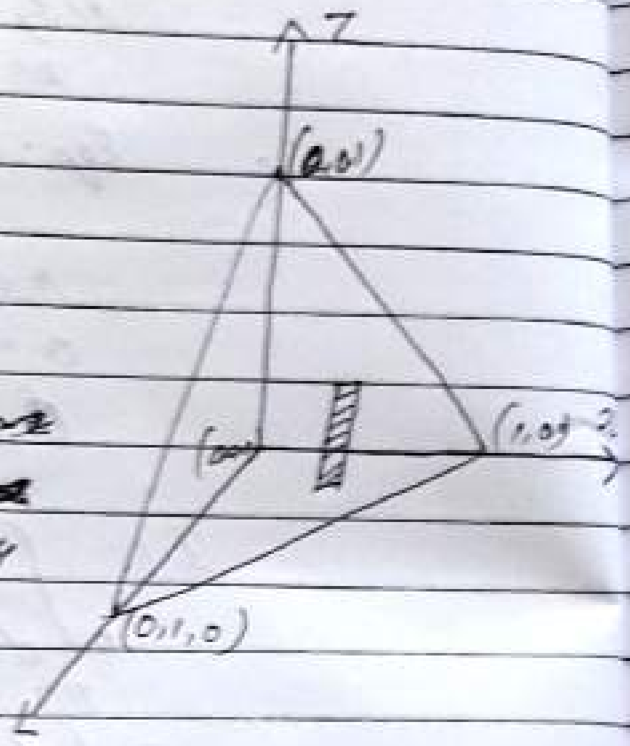
(13,14,15)

Ques ① Find the volume of tetrahedron bounded by $x=0, y=0, z=0$ and $x+y+z=a$

Sol.



$x=0$ to $x=a$
$y=0$ to $y=a-x$
$z=0$ to $z=a-x-y$



$$V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} (a-x-y)\,dy\,dx$$

Don't solve this seperately

$$V = \int_0^a \left[\frac{(a-x-y)^2}{-2}\right]_0^{a-x} dx$$

$$= \frac{-1}{2}\int_0^a \left[0 - (a-x)^2\right] dx$$

$$= \frac{1}{2}\left[\frac{(a-x)^3}{-3}\right]_0^a = \frac{a^3}{6} \text{ cubic units}$$